Practice Final Exam for Anaysis I, Math 4317, December 3 , 2010, Allowed time is 2 hours and 50 minutes. This is a closed book test. Always state your reasoning otherwise credit will not be given. Try to be as concise and to the point as possible.

Name:

1: The series
$$1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + - \cdots$$
is alternating. Does it converge?

2: Let $a_1 \geq a_2 \geq \dots$ be a decreasing series of positive real numbers. Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if the series
$$\sum_{n=0}^{\infty} 2^n a_{2^n}$$
converges.

3: Let E be a metric space and $A \subset E$ open. If $B \subset E$ is any other set then
$$A \cap \overline{B} \subset \overline{A \cap B} \ .$$

4: Consider the Dirichlet function $f : [0,1] \to \mathbb{R}$ given by
$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } p, q \text{ have no common divisor} \\ 0 & \text{if } x \text{ is irrational} \ . \end{cases}$$
For any $x_0 \in [0,1]$ compute $\lim_{x \to x_0} f(x)$.

5: A real valued function on a metric space E is **lower semicontinuous** if for all $t \in \mathbb{R}$, the set $\{p \in E : f(p) > t\}$ is open. Prove that a function f is lower semicontinuous if an only if for every sequence $p_1, p_2, p_3, \dots$ with $\lim_{n\to\infty} p_n = p$
$$\liminf_{n\to\infty} f(p_n) \geq f(p) \ .$$

6: Let $p_n(x)$ be a polynomial of degree n which only simple real roots $r_1 < r_2 < \cdots < r_n$. Show that $p_n'(x)$ is a polynomial of degree $n-1$ that has only simple real roots $s_1 < s_2 < \cdots < s_{n-1}$ as well and
$$r_1 < s_1 < r_2 < s_2 < \cdots < s_{n-1} < r_n \ .$$

7: Is the function
$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

differentiable everywhere?

8: Prove that a real–valued function on $\mathbb{R}$ with bounded derivative is uniformly continuous.

9: Does the integral $\int_0^1 f(x)dx$ exist for the function given in problem **5**?

10: Compute

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right) .$$

11: Show, by directly using the definition that

$$\lim_{x\to\infty}\frac{\log(x)}{x^{\varepsilon}}=0$$

for every $\varepsilon > 0$.

12: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for every $r > 0$ and $x \in \mathbb{R}$

$$\frac{1}{2r}\int_{x-r}^{x+r} f(t)dt = f(x) .$$

Show that the there exist constants a, b such that $f(x) = a + bx$.