Practice Final Exam for Anaysis I, Math 4317, December 3 , 2010, Allowed time is 2 hours and 50 minutes. This is a closed book test. Always state your reasoning otherwise credit will not be given. Try to be as concise and to the point as possible.

Name:

1: The series

$$1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + - \cdots$$

is alternating. Does it converge?

No it does not. The assumption that the terms are decreasing is violated and hence the theorem about alternating series is not applicable. To see that it diverges note that the series is of the form

$$\sum_{m=0}^{\infty} [\frac{1}{2m} - \frac{1}{(2m+1)^2}]$$

and

$$\frac{1}{2m} - \frac{1}{(2m+1)^2} \geq \frac{1}{2m+1} - \frac{1}{(2m+1)^2} = \frac{2m}{(2m+1)^2}$$

and the comparison test with the harmonic series shows that that the given series must be divergent.

2: Let $a_1 \geq a_2 \geq \dots$ be a decreasing series of positive real numbers. Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if the series

$$\sum_{n=0}^{\infty} 2^n a_{2^n}$$

converges.

Group the partial sums as follows

$$a_1 + (a_2 + a_3) + (a_4 + \cdots + a_7) + (a_8 + \cdots + a_{15}) + \cdots + (a_{2^n} + \cdots a_{2^{n+1}-1})$$

which is bounded above by

$$s_n = 2^0 a_{2^0} + 2^1 a_{2^1} + 2^2 a_{2^2} + 2^3 a_{2^3} + \cdots + 2^n a_{2^n}$$

Thus whenever the latter series converges so does the original one. For the lower bound we group things differently:

$$a_1 + a_2 + (a_3 + a_4) + (a_5 + \cdots a_8) + \cdots + (a_{2^{n-1}-1} + \cdots a_{2^n})$$

$$\geq a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{n-1}a_{2^n} \geq \frac{1}{2}s_n \ .$$

3: Let E be a metric space and $A \subset E$ open. If $B \subset E$ is any other set then

$$A \cap \overline{B} \subset \overline{A \cap B} \ .$$

Hence when the original series converges so does the one whose partial sums are given by s_n.

Let $p \in A \cap \overline{B}$ be any point. If $p \in A \cap B$ then $p \in \overline{A \cap B}$. If $p \notin A \cap B$ there exists a sequence of points $p_n \in B$ such that p_n converges to p. Since $p \in A \cap \overline{B}$ and A is open, there exists a ball of radius $\varepsilon > 0$ centered at p which is entirely contained in A. Hence all but finitely many of the points in the sequence p_n are in $A \cap B$. Since $\overline{A \cap B}$ is closed and p_n converges to p, $p \in \overline{A \cap B}$.

4: Consider the Dirichlet function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } p, q \text{ have no common divisor but one} \\ 0 & \text{if } x \text{ is irrational .} \end{cases}$$

For any $x_0 \in [0,1]$ compute $\lim_{x \to x_0} f(x)$.

Pick any positive integer Q and note that the only points where $f(x) \geq \frac{1}{Q}$ are

$$\frac{1}{Q}, \frac{2}{Q}, \cdots \frac{Q-1}{Q} \ , \frac{1}{Q-1}, \cdots \frac{Q-2}{Q-1} \ , \text{ etc.}$$

and hence we have

$$1 + (Q-1) + (Q-2) + \cdots + 2 + 1 = \frac{Q(Q-1)}{2} + 1$$

such points. The 1 counts the value at $x = 1$. Thus, for all but finitely many points we have that $f(x) < \frac{1}{Q}$. Thus, let x_n be any sequence that converges to x_0. Pick any $\varepsilon > 0$ and let Q be the smallest integer with

$$\frac{1}{Q} < \varepsilon \ .$$

Since there are only $\frac{Q(Q-1}{2}$ points where $f(x) \geq \frac{1}{Q}$ we find N so that for all $n > N$

$$f(x_n) < \frac{1}{Q} < \varepsilon$$

Hence, since $\varepsilon > 0$ is arbitrary,

$$\lim_{x \to x_0} f(x) = 0 \ .$$

5: A real valued function on a metric space E is **lower semicontinuous** if for all $t \in \mathbb{R}$, the set $\{p \in E : f(p) > t\}$ is open. Prove that a function f is lower semicontinuous if an only if for every sequence $p_1, p_2, p_3, \ldots$ with $\lim_{n\to\infty} p_n = p$

$$\liminf_{n\to\infty} f(p_n) \ge f(p) \ .$$

Assume first that

$$\liminf_{n\to\infty} f(p_n) \ge f(p) \ .$$

for every convergent sequence $p_n \in E$ with $\lim p_n = p$. Fix any t and consider the set

$$C_t = \{p \in E : f(p) \le t\} \ .$$

We want to show that this set is closed in E. Pick any sequence p_n in C_t which converges in E to some p. Since $t \ge f(p_n)$ for all n we have that

$$t \ge \liminf f(p_n) \ge f(p) \ .$$

Hence $p \in C_t$ and C_t is closed.

Now we assume that $\{p \in E : f(p) > t\}$ is open for all t. Let p_n be a convergent sequence with $\lim_{n\to\infty} p_n = p$ and assume that

$$\liminf f(p_n) = t \ .$$

Note, that when $\liminf f(p_n)$ is $+\infty$ there is nothing to prove. If it $-\infty$ then for any $M > 0$ there exists a subsequence, again denoted by p_n such that $f(p_n) \le -M$. Since the set $\{p \in E : f(p) \le -M\}$ is closed we have that $f(p) \le -M$. Since but $f(p)$ is defined and hence this situation is not possible. Thus we may assume that t is a finite number. Hence, for any $\varepsilon > 0$ we have

$$\liminf f(p_n) < t + \varepsilon \ .$$

Recall that $\liminf f(p_n)$ is the limit of the increasing sequence

$$a_k := \inf\{f(p_n) : n \ge k\}$$

Hence there exists a subsequence, again denoted by p_n such that $f(p_n) \le t + \varepsilon$. Since the set $\{p \in E : f(p) \le t + \varepsilon\}$ is closed and $p_n \to p$ we have that $f(p) \le t + \varepsilon$. Since $\varepsilon > 0$ is arbitrary $f(p) \le t$.

6: Let $p_n(x)$ be a polynomial of degree n which only simple real roots $r_1 < r_2 < \cdots < r_n$. Show that $p_n'(x)$ is a polynomial of degree $n-1$ that has only simple real roots $s_1 < s_2 < \cdots < s_{n-1}$ as well and

$$r_1 < s_1 < r_2 < s_2 < \cdots < s_{n-1} < r_n \ .$$

Pick any two successive points r_j and r_{j+1}. Since $p_n(r_j) = 0 = p_n(r_{j+1})$ there exists s_j with $r_j < s_j < r_{j+1}$ such that $p_n'(s_j) = 0$. This follows from the mean value theorem (or more properly Rolle's theorem). Thus, $p_n'(x)$ is a polynomial of degree $n-1$ and has $n-1$ roots $s_1, s_2, \dots, s_{n-1}$ which are sandwiched between successive numbers r_j and r_{j+1}. Hence the $s-j$ are all the roots of the polynomial $p_n'(x)$ and moreover they must be simple.

7: Is the function

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

differentiable everywhere?

Want to show that the function is not differentiable at $x = 0$. It suffices to pick one sequence h_n and to show that the limit

$$\lim_{n \to \infty} \frac{f(h_n) - f(0)}{h_n}$$

does not exist. Just pick $h_n = 1/n$ and compute

$$\frac{f(h_n) - f(0)}{h_n} = \sin n$$

which does not have a limit as $n \to \infty$.

8: Prove that a real–valued function on $\mathbb{R}$ with bounded derivative is uniformly continuous.

Call this function f and pick any two numbers $x, y \in \mathbb{R}$. Then by the mean value theorem there exists ξ between x and y such that

$$f(x) - f(y) = f'(\xi)(x - y) \ .$$

Since the derivative is bounded there exists a constant $M > 0$ such that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Hence

$$|f(x) - f(y)| \leq M|x - y|$$

for all $x, y \in \mathbb{R}$. Pick any $\varepsilon > 0$ and set $\delta = \frac{\varepsilon}{M}$. Then for any x, y with $|x - y| < \delta$ we have

$$|f(x) - f(y)| < M\delta = \varepsilon \ .$$

Hence the function is uniformly continuous.

9: Does the integral $\int_0^1 f(x)dx$ exist for the function given in problem **5**?

Pick any $\varepsilon > 0$ and pick Q the smallest positive integer such that $\frac{1}{Q} < \varepsilon/2$. Now we pick a partition. Denote the points where the function $f(x)$ is greater than $1/Q$ by $c_1, \dots, c_M$ where $M = \frac{Q(Q-1)}{2} + 1$. For each c_i consider the closed interval $[c_i - u\varepsilon/2, c_i + u\varepsilon/2]$ where we choose the number u later, and define the partition to consist of the points $c_i \pm u\varepsilon/2, i = 1 \ldots, M$. Now, the upper sum can be bounded above as follows: the contribution from the intervals $[c_i - u\varepsilon/2, c_i + u\varepsilon/2]$ is bounded above by $u\varepsilon$ times the largest value of the function which is smaller than 1. Hence the total contribution is $Mu\varepsilon$. From the other intervals the contribution is at most $\varepsilon/2$ since the function is smaller than $\varepsilon/2$ on the remaining set. This is independent of the choice of u.Hence the upper sum is bounded above by

$$Mu\varepsilon + \varepsilon/2$$

and if we choose $u = \frac{1}{2M}$ we get that the upper sum is bunded above by ε. Since the function is positive 0 is a lower bound on the lower sum and hence the function f is integrable and its integral is 0.

10: Compute

$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right) .$$

Here we factor out n and get

$$I_n = \frac{1}{n} \left(\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{2} \right)$$

Thus we see that this is related to a Riemann sum for the function

$$\frac{1}{1+x} .$$

For the uniform partition $x_n = 1 + \frac{k}{n}, k = 0, 1, \ldots, n$ consider the upper sum

$$U_n = \sum_{k=1}^{n} \frac{1}{1+x_{k-1}} \frac{1}{n} = \frac{1}{n} \left[1 + \frac{1}{1+\frac{1}{n}} \cdots \frac{1}{2-\frac{1}{n}} \right]$$

and the lower sum

$$L_n = \sum_{k=1}^{n} \frac{1}{1+x_k} \frac{1}{n} = \frac{1}{n} \left[\frac{1}{1+\frac{1}{n}} \cdots \frac{1}{2-\frac{1}{n}} + \frac{1}{2} \right]$$

Clearly

$$U_n = I_n - \frac{1}{2n} \, , L_n = I_n \ .$$

The integral

$$\int_0^1 \frac{1}{1+x} dx$$

exists and we have that

$$L_n \le \int_0^1 \frac{1}{1+x} dx \le U_n \ .$$

Moreover
$$U_n - L_n = \frac{1}{2n} \to 0$$
as $n \to \infty$. Hence
$$\lim_{n\to\infty} I_n = \int_0^1 \frac{1}{1+x} dx = \log(2) \ .$$
11: Show, by directly using the definition that
$$\lim_{x\to\infty} \frac{\log(x)}{x^\varepsilon} = 0$$
for every $\varepsilon > 0$.

We have that
$$\log(x) = \int_1^x \frac{1}{t} dt$$
Since for $t > 1$ we have that $\frac{1}{t} \le \frac{1}{t^{1-\beta}}$ for any $\beta > 0$ we find that
$$\log(x) = \int_1^x \frac{1}{t} dt \le \int_1^x \frac{1}{t^{1-\beta}} dt = \frac{1}{\beta} \left[x^\beta - 1 \right]$$
For any $\varepsilon > 0$ pick $\beta = \varepsilon/2$ and note that
$$\frac{\log(x)}{x^\varepsilon} \le \frac{2}{\varepsilon} \left[x^{-\varepsilon/2} - x^{-\varepsilon} \right]$$
which vanishes as $x \to \infty$.

12: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for every $r > 0$ and $x \in \mathbb{R}$
$$\frac{1}{2r} \int_{x-r}^{x+r} f(t) dt = f(x) \ .$$
Show that the there exist constants a, b such that $f(x) = a + bx$.

First note that since the left side is differentiable, so os f. Differentiating with respect to x we find
$$\frac{1}{2r} [f(x+r) - f(x-r)] = f'(x) \ .$$
Since f is differentiable, the left side is differentiable and hence f is twice continuously differentiable. Now write the equation as
$$\int_{x-r}^{x+r} f(t) dt = 2r f(x) \ .$$
and differentiate with respect to r:
$$f(x+r) + f(x-r) = 2f(x) \ .,$$

or

$$\frac{f(x+r)+f(x-r)-2f(x)}{r^2}=0\ .$$

Using Taylor's theorem the left side converges to $f''(x)$ as $r \to 0$. Hence $f''(x)=0$ and

$$f(x)=a+bx$$

for some constants a, b.