Test 2 for Anaysis I, Math 4317, November 5 , 2010, Allowed time is 50 minutes. This is a closed book test. Always state your reasoning otherwise credit will not be given. Try to be as concise and to the point as possible.

Name:

1: a) (10 points) For which values of A is the function $f(x)$ on $[-1,1]$ given by

$$f(x) = \begin{cases} (x+1)^2 & \text{if } 0 < x \le 1 \ , \\ x + A & \text{if } -1 \le x \le 0. \end{cases}$$

continuous.

Set $A = 1$.

b) (5 points) Is the function you got in a) uniformly continuous?

Yes, every function continuous on a compact set is uniformly continuous there. The interval $[-1,1]$ is compact.

2: (15 points) A function $f : \mathbb{R} \to \mathbb{R}$ is called uniformly Hölder-continuous of order $\alpha > 0$ if there exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}$

$$|f(x) - f(y)| \le C|x-y|^\alpha \ .$$

Show that such a function is continuous.

For any given $\varepsilon > 0$ pick any δ such that

$$C\delta^\alpha < \varepsilon \ ,$$

or any δ such that

$$\delta < \left(\frac{\varepsilon}{C}\right)^{\frac{1}{\alpha}} \ .$$

will do.

3: (5 points) For $n = 1, 2, 3, \ldots$ consider the functions $f_n : \mathbb{R} \to \mathbb{R}$ given by

$$f_n(x) = \frac{1}{1 + (x - n)^2} \ .$$

a) Does this sequence of functions converge? If yes, what is the limiting function.

Yes this sequence of functions converges to the zero function.

b) (5 points) Is the convergence uniform?

It does NOT go uniformly to zero. Just 'follow' the sequence by setting $x = n$, i.e., $f(n) = 1$ which does not tend to zero.

4: Let E, E' be two metric spaces and let $f : E \to E'$ be a continuous function.
a) (10 points) Prove that for every closed set $S \subset E'$. $f^{-1}(S)$ is closed in E.

S closed means that S^c is open. Now $f^{-1}(S^c)$ is the set of all points in E whose image under f is not in S, which is the complement of the set of all points in E whose image under f is in S. Hence

$$f^{-1}(S^c) = \left(f^{-1}(S)\right)^c$$

Since f is continuous, $f^{-1}(S^c)$ is open and hence $f^{-1}(S)$ is closed.

Assume in addition that E is compact and that f is one-to-one and onto.
b) (20 points) Let q_n be a sequence in E' that converges to $q \in E'$. Prove that $p_n := f^{-1}(q_n)$ converges to $p := f^{-1}(q)$ in E. (Hint: Use the fact that if a sequence has the property that every convergent subsequence has the same limit, then the sequence converges.)

Since E is compact, the sequence p_n has a convergent subsequence p_{n_k} with limit, say p. Since f is continuous

$$f(p) = \lim_{k \to \infty} f(p_{n_k}) = \lim_{k \to \infty} q_{n_k} = q \ .$$

Thus, $p = f^{-1}(q)$. The same argument shows that any convergent subsequence of p_n converges to p. And hence because of the hint, the whole sequence p_n converges to p.

If a sequence has the property that every convergent subsequence converges to the same limit, then the whole sequence converges. To see this, assume that it does not converge. Thus, there exists $\varepsilon > 0$ and a subsequence p_{n_k} such that

$$d(p, p_{n_k}) > \varepsilon$$

for all k. Since E is compact, there is a further subsequence that converges, and since it is a subsequence of the original sequence it must converge to p. This contradicts the fact that $d(p, p_{n_k}) > \varepsilon$ for all k.

c) (5 points) What can you conclude from b) about the inverse function $f^{-1}(q)$, $q \in E'$?

The inverse function is continuous.

5: True or false: (5 points each)

a) Every real function $f(x, y)$ on E^2 which, for every fixed x, is continuous as a function of y and which, for every fixed y, is continuous as a function of x, is continuous as a function from $E^2 \to \mathbb{R}$.

FALSE

Problem 10b) on page 92 of IA provides a counterexample.

b) A continuous function $f : E \to E'$ where E, E' are metric space has the property that for any open set $S \subset E$, $f(S) \subset E'$ is also open.

FALSE

Take the continuous function f from the reals to the reals that assigns to every number the constant 1. Then $f(\mathbb{R}) = \{1\}$ which is not open.

c) Any convergent sequence of continuous functions defined on a compact metric space converges uniformly.

FALSE

x^n on $[0, 1]$ furnishes a counterexample.

d) A continuous function defined on a compact metric space is uniformly continuous.

TRUE

This was proved in class.

e) If $f : E \to E'$ is continuous and if E is compact, then $f(E)$ is also compact.

TRUE

Was also proved in class.

Additional credit: (15 points) Let $f : E \to E'$ be a function. Prove that the function is continuous if and only if for any subset $S \subset E'$

$$f^{-1}(S^\circ) = \left\{f^{-1}(S)\right\}^\circ$$

where S° is the open interior of S. Recall that

$$S^\circ = \cup_{U \subset S, open} U \ ,$$

i.e., the largest open set that is a subset of S.

The problem as it is written is not correctly posed. Again, take the constant function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1$ all $x \in \mathbb{R}$. Then $f^{-1}(\{1\}) = \mathbb{R}$ and hence $f^{-1}(\{1\})^\circ = \mathbb{R}$. But $\{1\}^\circ = \emptyset$ and hence $f^{-1}(\{1\}^\circ) = \emptyset$. What is, true, however, is that

$$f^{-1}(S^\circ) \subset \left\{f^{-1}(S)\right\}^\circ \tag{0.1}$$

holds for every set $S \subset E$ if and only if f is continuous. Assume now that 0.1 holds for all sets $S \subset E$. Then it holds for all open sets $S \subset E$ and hence since $S^\circ = S$

$$f^{-1}(S) \subset \left\{f^{-1}(S)\right\}^\circ \subset f^{-1}(S)$$

and hence

$$f^{-1}(S) = f^{-1}(S)^\circ$$

and $f^{-1}(S)$ is open.

Conversely, assume that f is continuous. We have that $S^\circ \subset S$ and S° is open. Hence $f^{-1}(S^\circ) \subset f^{-1}(S)$ and $f^{-1}(S^\circ)$ is open. Since $f^{-1}(S)^\circ$ is the union of all open subsets of $f^{-1}(S)$, we have $f^{-1}(S^\circ) \subset f^{-1}(S)^\circ$.