Homework Set for Week 12

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Chapter 5

5. **Assuming the elementary properties of the trigonometric functions, show that $\tan x - x$ is strictly increasing on $(0, \frac{\pi}{2})$ while the function $\frac{\sin x}{x}$ is strictly decreasing.**

   Solution
   From problem 4 (page 109 of IA) we get:

   Lemma: if f is a differentiable real-valued function on an open interval in $\mathbb{R}$ then f is increasing (decreasing) if f' is nonnegative (nonpositive) at each point in the interval.

   Proof of the Lemma: Let $f : [a, b] \to \mathbb{R}$ be differentiable on (a, b). Let $x, y \in (a, b)$ and let $y > x$. By the Mean Value Theorem (page 105 of IA) there exists $c \in (x, y)$ such that

   $$f(y) - f(x) = f'(c)\,(x - y)$$

   If $f'(c) \geq 0$ since $y > x$ we get $f(y) - f(x) = f'(c)\,(x - y) \geq 0$, which implies $f(y) \geq f(x)$, i.e. f is increasing. Similarly, if $f'(c) \leq 0$ since $y > x$ we get $f(y) - f(x) = f'(c)\,(x - y) \leq 0$, which implies $f(y) \leq f(x)$, i.e. f is decreasing. Since this holds for all x, y then it is true for the entire interval (a, b). Furthermore, if we switch the inequalities from $\geq$ ($\leq$) to $>$ ($<$) it follows that f is strictly increasing (decreasing).

   □

   To show: $f : (0, \frac{\pi}{2}) \to \mathbb{R}$, where $f(x) = \tan x - x$ is strictly increasing by showing $f'(x) > 0$ for $x \in (0, \frac{\pi}{2})$ and $g : (0, \frac{\pi}{2}) \to \mathbb{R}$, where $g(x) = \frac{\sin x}{x}$ is strictly decreasing by showing $g'(x) < 0$ for $x \in (0, \frac{\pi}{2})$

   We know by the proposition in page 101 of IA and by the properties of the trigonometric properties that

   $$f'(x) = \sec^2 x - 1 > 0 \quad \text{for all } x \in \left(0, \frac{\pi}{2}\right)$$

   and hence $f(x) = \tan x - x$ is strictly increasing. Similarly,

   $$g'(x) = \frac{x\cos x - \sin x}{x^2} \cdot \frac{\cos x}{\cos x} = -\frac{\cos x}{x^2}\,(\tan x - x)\,.$$

   Since we just showed that $\tan x - x$ is strictly increasing in this interval and $\tan(0) = 0$, then $\tan x - x > 0$. We also know that $x^2 > 0, \cos x > 0$ on the interval $x \in (0, \frac{\pi}{2})$. Hence $g'(x) < 0$ on $x \in (0, \frac{\pi}{2})$ and therefore $g(x) = \frac{\sin x}{x}$ is strictly decreasing.

6. Prove that a differentiable function on $\mathbb{R}$ with a bounded derivative is uniformly continuous.

   <u>*Proof*</u>
   Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with bounded derivative, i.e. there exists $M \in \mathbb{R}$, $M \geq 0$ so that

   $$|f'(x)| \leq M \quad \text{for all } x \in \mathbb{R}.$$

To show: For every $\epsilon > 0$ there exists δ so that whenever $|x - y| < \delta$ we have that $|f(x) - f(y)| < \epsilon$ for all $x, y \in \mathbb{R}$.

Select any $x, y \in \mathbb{R}$ and without loss of generality let $y > x$ (otherwise just interchange the two). Then by the Mean Value Theorem on page 105 of IA there exists $c \in (x, y)$ so that

$$\left| \frac{f(x) - f(y)}{x - y} \right| = |f'(c)| \qquad \text{or} \qquad |f(x) - f(y)| = |f'(c)|\,|x - y|\,.$$

But since $|f'|$ is bounded we get

$$|f(x) - f(y)| = |f'(c)|\,|x - y| \le M\,|x - y|$$

Now for any given $\epsilon > 0$ let $\delta = \frac{\epsilon}{M}$, then whenever $|x - y| < \delta$ we get that

$$|f(x) - f(y)| \le M\,|x - y| < M \cdot \frac{\epsilon}{M} = \epsilon.$$

Since δ here does not depend on x or y, uniform continuity follows.

□

8. Let $a, b \in \mathbb{R}, a < b$, and let f, g be continuous real-valued functions on $[a, b]$ that are differentiable on (a, b). Prove that there exists a number $c \in (a, b)$ such that

$$f'(c)\,(g(b) - g(a)) = g'(c)\,(f(b) - f(a))\,.$$

(*Hint:* Consider the function

$$F(x) = (f(x) - f(a))\,(g(b) - g(a)) - (g(x) - g(a))\,(f(b) - f(a))\,.)$$

<u>*Proof*</u>

We consider the function $F(x)$ as stated above. Since both f and g are differentiable on (a, b), then by the proposition on page 101 of IA the derivative of $F(x)$ is given by

$$F'(x) = f'(x)\,(g(b) - g(a)) - g'(x)\,(f(b) - f(a))\,.$$

Since $F(a) = F(b) = 0$, continuous and real-valued on $[a, b]$ and differentiable on (a, b) by Rolle's theorem (page 104 of IA) there exists a $c \in \mathbb{R}$ so that $F'(c) = 0$. Hence we find

$$F'(c) = 0 = f'(c)\,(g(b) - g(a)) - g'(c)\,(f(b) - f(a))$$

or

$$f'(c)\,(g(b) - g(a)) = g'(c)\,(f(b) - f(a))$$

□

Chapter VI

2. Prove that $\int_0^1 f(x)\,\mathrm{d}x = 0$ if $f(1/n) = 1$ for $n = 1, 2, 3, \ldots$ and $f(x) = 0$ for all other x.

   Proof

   Let the partition sequence $\mathcal{P}_n$ be defined as

$$x_k = \frac{k}{n}, \quad k = 0, 1, 2, \ldots, n$$

   where $n \in \{x \in \mathbb{N} : x > 3 \text{ and } x \text{ is a prime number}\}$. With this sequence of partitions the upper sum is given by

$$\begin{aligned} U_f(\mathcal{P}_n) &= \sum_{j=0}^{n-1} \ell.u.b.\{f(x) : x_j \leq x \leq x_{j+1}\}(x_{j+1} - x_j) \\ &= (x_1 - x_0) + (x_2 - x_1) + (x_n - x_{n-1}) = \frac{3}{n} \end{aligned}$$

   whereas the lower sum is given by

$$L_f(\mathcal{P}_n) = \sum_{j=0}^{n-1} g.\ell.b.\{f(x) : x_j \leq x \leq x_{j+1}\}(x_{j+1} - x_j) = 0$$

   since $f(1/n) = 1$ and $f(x) = 0$ for all other x. It is then made clear that

$$U_f(\mathcal{P}_n) - L_f(\mathcal{P}_n) = \frac{3}{n} \to 0 \quad \text{as} \quad n \to \infty$$

   and thus the integral exists. Furthermore since we can always select $n > N$ for any $\epsilon > 0$ so that $\epsilon > \frac{3}{n}$, and consequently $\epsilon > U_f(\mathcal{P}_n) \geq I_U(f) \geq I_L(f) \geq L_f(\mathcal{P}_n) \geq 0$. It follows that $\int_0^1 f(x)\,\mathrm{d}x = 0$

   □

3. Does $\int_0^1 f(x)\,\mathrm{d}x$ exist if f is defined as follows?

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is not rational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q}, \text{ where } p \text{ and } q \text{ are integers} \\ & \quad \text{with no common divisors} \\ & \quad \text{other than } \pm 1, \text{ and } q > 0. \end{cases}$$

   Solution

   Yes. Let $\mathcal{P}$ be a partition of $[0, 1]$. Then every interval $[x_j, x_{j+1}]$ contains both rational and irrational numbers. As such, the lower sum will be

$$L_f(\mathcal{P}) = \sum_{j=0}^{n-1} \inf_{x \in [x_j, x_{j+1}]} f(x) \ (x_{j+1} - x_j) = 0$$

since each partition contains an irrational number.

The upper sum is a little trickier to find. We use the fact that for any given $n \in \mathbb{N}$ there are only a finite number of x such that $f(x) \geq 1/n$. This is true since $f(x) = 0$ if x is not rational and if x is rational then $f(p/q) = 1/q \geq 1/n$ which implies that $0 < q \leq n$. Since $p/q \leq 1$ and p and q have no common divisors, there are at most n choices for p. Thus we may conclude that most points are close to zero and we will use this fact to upper bound the upper sum.

We know that there are at most m values for $x \in [0,1]$ so that $f(x) \geq 1/n$. Let $\{x_1, x_2, \ldots, x_m\}$ be a finite set of x values so that $f(x) \geq 1/n$ and let $M = max\{f(x) : x \in \{x_1, x_2, \ldots, x_m\}\}$. Now let $\epsilon/2 > 0$ and select n such that $1/n \leq \epsilon/2$. Finally, let

$$g(x) = \begin{cases} M & \text{if } x \in \{x_1, x_2, \ldots, x_m\} \\ 0 & \text{otherwise} \end{cases}$$

then $f(x) \leq \epsilon/2 + g(x)$. For any partition $\mathcal{P}$ of $[0,1]$ the upper sum is bounded by:

$$0 \leq U_f(\mathcal{P}) \leq U_{\epsilon+g}(\mathcal{P}) \leq U_\epsilon(\mathcal{P}) + U_g(\mathcal{P}) = \frac{\epsilon}{2} + U_g(\mathcal{P})$$

But $g(x)$ is continuous at 0 with a finite number of discontinuities, hence it is integrable with integral 0 (the proof to this will be shown in the following problem). As a consequence we can find a partition Q so that $U_g(\mathrm{Q}) < \epsilon/2$. It follows that

$$0 \leq L_f(\mathrm{Q}) \leq I_L(f) \leq I_U(f) \leq U_f(\mathrm{Q}) \leq \frac{\epsilon}{2} + U_g(\mathrm{Q}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and since ϵ can be made arbitrarily small the integral of f exists and $\int_0^1 f(x)\,dx = 0$.

□

7. Prove that if the real-valued function f on the interval $[a,b]$ is bounded and is continuous except at a finite number of points, then $\int_0^1 f(x)\,dx$ exists.

Proof

We first reduce the problem to the case of having exactly one discontinuity by breaking up the interval $[a,b]$ to subintervals where each subinterval contains exactly one discontinuous point. If f is integrable on each subinterval, then it is integrable on $[a,b]$ by the proposition on page 123 of IA.

We let $x^\star$ be the point of discontinuity of f on the subinterval $[a^\star, b^\star]$ and suppose $x^\star \neq a^\star$ and $x^\star \neq b^\star$. Then we select a smaller subinterval such that $x^\star \in (a_1, b_1) \subset [a^\star, b^\star]$ which satisfies

$$\left(\sup_{x \in [a^\star, b^\star]} f(x) - \inf_{x \in [a^\star, b^\star]} f(x) \right) (b_1 - a_1) < \frac{\epsilon}{2}$$

Then $[a^\star, b^\star] - (a_1, b_1)$ consists of two disjoint intervals where f is continuous and thus integrable by the theorem on page 123 of IA. We can find a combined partition $\mathcal{P}$

such that in total these two disjoints intervals result in $U_f(\mathcal{P}) - L_f(\mathcal{P}) < \epsilon/2$. But the points of $\mathcal{P}$ form a partition of all $[a^\star, b^\star]$, and since we selected the subinterval (a_1, b_1) to satisfy the upper and lower sum subtraction be less than $\epsilon/2$ we get

$$U_f(\mathcal{P}) - L_f(\mathcal{P}) < \epsilon.$$

Since a similar argument can be obtained if $x^\star = a^\star$ or $x^\star = b^\star$ we may conclude the integral exists.

□

7 $f: [a,b] \to \mathbb{R}$ continuous and bounded everywhere but at a finitely many number of points. Show that $\int_a^b f(x)dx$ exists

define x_i $i = 1, 2, \ldots, n$ discontinuities of $f(x)$

define $f_i(x) \doteq \begin{cases} f(x) & \text{if } i=0: a \le x < x_1;\ x_i < x < x_{i+1}\ i \ne 0, n \quad i=n: x_i < x \le b \\ \lim_{x \to x_{i+1}^-} f(x) & \text{if } x = x_{i+1} \\ \lim_{x \to x_i^+} f(x) & \text{if } x = x_i \\ 0 & x \notin [x_i, x_{i+1}] \end{cases}$

Then $f(x) - \sum_{i=0}^{n} f_i(x) = g(x)$ $\qquad g: [a,b] \to \mathbb{R}$

$g(x) = \begin{cases} 0 & \text{if } x \ne x_i \\ f(x) - f_{i-1}(x) - f_i(x) & \text{if } x = x_i \end{cases}$

then: $\int_a^b f(x) - \sum_{i=0}^{n} f_i(x) = \int_a^b g(x) = 0$

$\int_a^b f(x) = \int_a^b \sum_{i=0}^{n} f_i(x) \Rightarrow$ exists because these are bounded, continuous functions on compact space. (§3, p.123 IA)

To show: $\int_a^b g(x) = 0$

Any Rieman sum S corresponding to a partition of $[a,b]$ with width less than δ, we have $|S| < 2|f(x) - f_{i-1}(x) - f(x)|\delta$ so we choose δ sufficiently small so that $\int_a^b g(x) = 0$