Dylan McKay
Michael Walker
Jesse Wells

Math 4317: Homework Week 13
Solutions

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16 Prove that the real-valued function on $C([a,b])$ which sends any f into $\int_a^b f(x)dx$ is uniformly continuous.

Proof: Suppose V is a function such that $V(f) = \int_a^b f(x)dx$, where $f \in C([a,b])$. Let $f, g \in C([a,b])$ and $d(f(x), g(x)) < \delta$ for all $x \in [a,b]$.

$$d'(V(f(x)), V(g(x))) = \left|\int_a^b f(x)dx - \int_a^b g(x)dx\right| = \left|\int_a^b (f(x) - g(x))dx\right| < \delta(b-a) = \epsilon$$

$\delta > 0$, thus $\epsilon > 0$, therefore V is uniformly continuous. $\square$

17 Prove that if u and v are real-valued functions on an open subset of $\mathbf{R}$ containing the interval $[a, b$ and if u and v have continuous derivatives, then (integration by parts)

$$\int_a^b u(x)v'(x) = u(b)v(b) - u(a)v(a) - \int_a^b v(x)u'(x)dx.$$

Proof: Let $G(x) = u(x)v(x)$, then $G'(x) = u'(x)v(x) + u(x)v'(x)$ by the product rule. $\int_a^b G'(x)dx = G(b) - G(a) = u(b)v(b) - u(a)v(a)$.

$$\begin{aligned}
&\int_a^b u(x)v'(x)dx \\
&= \int_a^b G'(x) - u'(x)v(x)dx \\
&= \int_a^b G'(x)dx - \int_a^b v(x)u'(x)dx \\
&= u(b)v(b) - u(a)v(a) - \int_a^b v(x)u'(x)dx
\end{aligned}$$

$\square$

(b) $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}\right)$

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}\right)=\lim_{n\to\infty}\sum_{i=0}^{n}\frac{1}{n+1}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n}\frac{1}{1+\frac{i}{n}}$$

Let $f(x)=\frac{1}{1+x}$ and let $P=\{\frac{0}{n},\frac{1}{n},\frac{2}{n},...,\frac{n}{n}\}$ be a partition.

$$U_f(P)=\sum_{i=0}^{n-1}\left(\frac{1}{1+\frac{i}{n}}\right)\left(\frac{i+1}{n}-\frac{i}{n}\right)=\sum_{i=0}^{n-1}\left(\frac{1}{1+\frac{i}{n}}\right)\left(\frac{1}{n}\right)=\frac{1}{n}+\sum_{i=1}^{n}\left(\frac{1}{1+\frac{i}{n}}\right)\left(\frac{1}{n}\right)$$

$$L_f(P)=\sum_{i=0}^{n-1}\left(\frac{1}{1+\frac{i+1}{n}}\right)\left(\frac{i+1}{n}-\frac{i}{n}\right)=\sum_{i=1}^{n}\left(\frac{1}{1+\frac{i}{n}}\right)\left(\frac{1}{n}\right)$$

$$\lim_{n\to\infty}U_f(P)=\lim_{n\to\infty}\left(\frac{1}{n}+\sum_{i=1}^{n}\left(\frac{1}{1+\frac{i}{n}}\right)\left(\frac{1}{n}\right)\right)=\lim_{n\to\infty}\frac{1}{n}+\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{1+\frac{i}{n}}\right)\left(\frac{1}{n}\right)=$$
$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{1+\frac{i}{n}}\right)\left(\frac{1}{n}\right)=L_f(P)$$

Therefore, $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{1+\frac{i}{n}}\right)\left(\frac{1}{n}\right)=\int_0^1 f(x)dx=\int_0^1\frac{1}{1+x}dx=\log(1+x)\Big|_0^1=$ $\log(2)$

23 Prove that the only function $f:\mathbf{R}\to\mathbf{R}$ such that $f'=f$ and $f(0)=1$ is given by $f(x)=e^x$.

Proof: For every function $f(x)$, there exists a function $g(x)$ such that $f(x)=g(x)e^x$. By the product rule, $f'(x)=g'(x)e^x+g(x)e^x$. Since $f'(x)=f(x)$, $g'(x)e^x=0$, so $g'(x)=0$. If $g'(x)=0$, then $g(x)=M$ where $M\in\mathbf{R}$. $f(0)=f'(0)=g(0)e^0=Me^0=M=1$, so $g(x)=1$. Therefore, $f(x)=e^x$. □

(29) Prove that

a) $\log(1+x) \le x$ for all $x > -1$, with equality i.f.f. $x = 0$.

If $\log(1+x) \le x$ $\forall x > -1$, then $0 \le x - \log(x+1)$. Assume this is false. Then $\exists\, x > -1$ such that $x - \log(x+1) < 0$. Then, $x < \log(x+1)$ i.f.f. $e^x < x+1$ i.f.f. $e^x - x < 1$ iff, $\ln(e^x - x) < \ln 1 = 0$. But $\nexists\, y \in \mathbb{R}$ such that $\ln(y) < 0$. Contradiction. Thus $\log(1+x) \le x$ $\forall x > -1$. Let $\log(1+x) = x$. Then $1+x = e^x$. Taking the derivative of both sides, then $1 = e^x$ iff. $x = 0$. Now Let $x = 0$. Then since $\log(x+1) \le x$ $\forall x > -1$, Then $\log(1) \le 0$. But as before, $\nexists\, y \in \mathbb{R}$ such that $\log y < 0$, so $\log(1) = 0$. ■

b) $e^x \ge 1+x$ $\forall x$, with equality iff. $x = 0$

If $e^x \ge 1+x$ $\forall x$, then $e^x - 1 - x \ge 0$ $\forall x \in \mathbb{R}$. Assume this is false. Then $\exists\, x \in \mathbb{R}$ such that $e^x - 1 - x < 0$. Then $e^x - x < 1$ iff. $\ln(e^x - x) < \ln(1)$ i.f.f. $\ln(e^x - x) < 0$. But $\nexists\, y$ such $\ln(y) < 0$. So $\ln(e^x - x) \nless 0$ $\forall x$. Thus $e^x \ge 1+x$ $\forall x$. If $x = 0$, $e^0 = 1 \ge 1+0 = 1$. So $e^x = 1+x$ if $x = 0$. If $e^x = 1+x$ then $e^x - x = 1$ i.f.f. $e^x - 1 = 0$ iff. $e^x = 1$ i.f.f. $x = 0$. ■

c) $\lim\limits_{x \to 0} \dfrac{\log(1+x)}{x} = 1$

We know $\log x = \int_1^x \frac{1}{t}\,dt$, then $\log(1+x) = \int_0^x \frac{dt}{1+t}$

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = \lim_{x \to 0} \frac{1}{x} \int_0^x \frac{dt}{1+t}$$

Let $t = xu$, $dt = x\,du$, then $\lim\limits_{x \to 0} \frac{1}{x} \int_0^x \frac{dt}{1+t} = \lim\limits_{x \to 0} \int_0^1 \frac{du}{1+xu}$

$$= \int_0^1 \lim_{x \to 0} \frac{1}{1+xu}\,du = \int_0^1 du = 1$$

■

d) $\lim_{h\to 0} (1+xh)^{1/h} = \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$

We know $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

let $x \geq 0$, and define $s_n = \sum_{k=0}^{n} \frac{x^k}{k!}$, $t_n = \left(1+\frac{x}{n}\right)^n$

by the binomial theorem $t_n = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{x}{n}\right)^k = 1 + x + \frac{x^2}{2!}\left(1-\frac{1}{n}\right) + \frac{x^3}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) +$

$\cdots + \frac{x^n}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right) \leq \sum_{k=0}^{n} \frac{x^k}{k!} = s_n$

So $\lim_{n\to\infty} \sup t_n \leq \lim_{n\to\infty} \sup s_n = e^x$

Now, if $2 \leq m \leq n$, then

$1 + x + \frac{x^2}{2!}\left(1-\frac{1}{n}\right) + \cdots + \frac{x^m}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right) \leq t_n$

Fix m, then $s_m = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^m}{m!} \leq \lim_{n\to\infty} \inf t_n$

Then $\lim_{m\to\infty} s_m = e^x \leq \lim_{n\to\infty} \inf t_n$

So $\lim_{n\to\infty} \sup t_n \leq e^x \leq \lim_{n\to\infty} \inf t_n$ and thus

$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$

This means $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\left|\left(1+\frac{x}{n}\right)^n - e^x\right| < \varepsilon$ whenever $n > \frac{1}{\delta}$. Replace n by $\frac{1}{h}$ to obtain

$\left|(1+hx)^{1/h} - e^x\right| < \varepsilon$ whenever $\frac{1}{h} > \frac{1}{\delta}$ i.e. $\delta > h > 0$

which means $\lim_{h\to 0} (1+hx)^{1/h} = e^x = \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n$

e) $\lim_{n\to\infty} n(x^{1/n}-1) = \log x \quad \text{if } x>0$

Consider $\exp\left[\lim_{n\to\infty} n(x^{1/n}-1)\right] = \lim_{n\to\infty} \exp\left[n(x^{1/n}-1)\right] = \lim_{n\to\infty} \frac{e^{nx^{1/n}}}{e^{n}}$

$= \lim_{n\to\infty} \frac{\left(e^{x^{1/n}}\right)^n}{e^{n}} = \lim_{n\to\infty} \left(\frac{e^{x^{1/n}}}{e}\right)^n = \lim_{n\to\infty} \left(\frac{\log(e^{x^{1/n}})}{\log(e)}\right)^n = \lim_{n\to\infty} \left(x^{1/n}\right)^n$

$= \lim_{n\to\infty} x = x$ which is true iff $\lim_{n\to\infty} n(x^{1/n}-1) = \log x$

■