MATH 4317 Homework 12

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Contents

Problem 9 2

Problem 11 3

Problem 12 4

Problem 13 5

Problem 15 6

Problem 9

(Integral test). Let $f : \{x \in \mathbb{R} | x \geqslant 1\} \to \mathbb{R}$ be a decreasing positive-valued function. Prove that $\displaystyle\sum_{n=1}^{\infty} f(n)$ converges if and only if $\displaystyle\lim_{n\to\infty} \int_1^n f(x)\mathrm{d}x$ exists. (Hint: draw a diagram.)

Let $f : \{x \in \mathbb{R} | x \geqslant 1\} \to \mathbb{R}$ be a decreasing positive valued function. Let $S_n = \sum_{j=1}^{n} f(n)$. f is bounded, since $0 < f(x) \leqslant f(1)$ for all x, and monotone, so it is integrable on any closed interval $[a, b]$ with $a \geqslant 1$. In particular $\int_1^n f(x)\mathrm{d}x$ exists for any positive integer n. Define $\mathcal{P}_n = \{1, 2, \ldots, n\}$ as a partition of $[1, n]$, then $U_f(\mathcal{P}) = \sum_{j=1}^{n-1} = S_{n-1}$ and $L_f(\mathcal{P}) = \sum_{j=1}^{n-1} f(j+1) = S_n - f(1)$. Since $0 \leqslant L_f(\mathcal{P}) \leqslant \int_1^n f(x)\mathrm{d}x \leqslant U_f(\mathcal{P})$, it follows that $0 \leqslant S_n - f(1) \leqslant \int_1^n f(x)\mathrm{d}x \leqslant S_{n-1}$.

Suppose $\sum_{n=1}^{\infty} f(n)$ converges. Denote its limit by L, then $\lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} S_n = L$ and so $\int_1^n f(x)\mathrm{d}x \leqslant S_{n-1} \leqslant L$ all n, hence $\lim_{n\to\infty} \int_1^n f(x)$ exists.

Suppose $\lim_{n\to\infty} \int_1^n f(x)\mathrm{d}x$ exists. Denote its value by I, then $S_n - f(1) \leqslant \int_1^n f(x)\mathrm{d}x \leqslant I$ and so $S_n \leqslant I + f(1)$. S_n is bounded and monotonically increasing (since $f(n) > 0$ all n), and so it converges. $\square$

Problem 11

Show the convergence of the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+x}\right)$$

of real-valued functions on $\mathbb{R}\setminus\{-1,-2,-3,\dots\}$.

Let $f_1, f_2, f_3, \dots$ be a sequence of real-valued functions on $\mathbb{R}\setminus\{-1,-2,-3,\dots\}$ with $f_n(x) = \sum_{j=1}^{n}\left(\frac{1}{j}-\frac{1}{j+x}\right)$. We show that the sequence of functions converges pointwise. There are three cases to consider:

If $x = 0$, then $f_n(x) = 0$ for all n, and so clearly $\lim_{n\to\infty} f_n(x) = 0$.

If $x > 0$, then we write $f_n(x) = \sum_{j=1}\left(\frac{1}{j}-\frac{1}{j+x}\right) = \sum_{j=1}\frac{x}{j(j+x} = x\sum_{j=1}^{n}\frac{1}{j(j+x)}$. Choose N so that $N > x$, then for all $j > N$,we have $\frac{1}{j(j+x)} \leqslant \frac{1}{j(j+j)} = \frac{1}{2j^2}$. We know that $\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j^2}$ exists (in fact the limit is $\frac{\pi^2}{6}$), and so $\lim_{n\to\infty}\frac{x}{2}\sum_{j=1}^{n}\frac{1}{j^2}$ exists and by the comparison test, $\lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{1}{j}-\frac{1}{j+x}\right)$ exists.

If $x < 0$, choose N so that $N > |x|$. Then $\frac{1}{j(j+x)} \leqslant \frac{1}{j(j+0)} = \frac{1}{j^2}$ for all $j > N$, and so similarly $\lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{1}{j}-\frac{1}{j+x}\right)$ exists by the comparison test. $\square$

Problem 12

Show that if $a_1 + a_2 + a_3 + \dots$ is an absolutely convergent series of real numbers, then $a_1^2 + a_2^2 + a_3^2 + \dots$ converges.

Since $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, $\sum_{n=1}^{\infty} |a_n|$ converges. Therefore $\lim_{n\to\infty} a_n = 0$. Convergent sequences are bounded, so there exists $M \in \mathbb{R}$ so that $|a_n| \leqslant M$ for all n, and so $a_n^2 = |a_n||a_n| \leqslant M|a_n|$ for all n. Since $\sum_{n=1}^{\infty} |a_j|$ converges, $\sum_{n=1}^{\infty} M|a_n|$ converges, and so by the comparison test, $\sum_{n=1}^{\infty} a_n^2$ converges. $\square$

Problem 13

(Root test). Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers. Show that if there exists a number $\rho < 1$ such that $\sqrt[n]{|a_n|} \leqslant \rho$ for all sufficiently large n, then the series is absolutely convergent.

Let $\sum_{n=1}^{\infty} a_n$ be a series where $|a_n|^{\frac{1}{n}} \leqslant \rho$ for all $n > N$ for some N, where $\rho < 1$. Then $|a_n| \leqslant \rho^n$ for all $n > N$. Since $|a_n| \geqslant 0$, this means $|\rho| < 1$, and so the series $\sum_{n=1}^{\infty} \rho^n$ converges. Therefore, the series $\sum_{n=1}^{\infty} a_n$ converges by the comparison test. $\square$

Problem 15

Prove that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent series of real numbers then the series $\sum_{n,m=1}^{\infty} a_n b_m$ is also absolutely convergent, and

$$\sum_{n,m=1}^{\infty} a_n b_m = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right).$$

Since $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$. Convergent sequences are bounded, so there exists $M \in \mathbb{R}$ so that $|a_n| \leqslant M$ and $|b_n| \leqslant M$ for all n. Hence $|a_n|b_m \leqslant |a_n|M$ for all m. As M is constant, $\sum_{n=1}^{\infty} a_n M$ converges absolutely, so by the comparison test, $\sum_{n,m=1}^{\infty} a_n b_m$ converges absolutely. Therefore any rearrangement of its terms converges absolutely (to the same limit). Hence,

$$\begin{aligned}\sum_{n,m=1}^{\infty} a_n b_m &= (a_1b_1 + a_1b_2 + a_1b_3 + \dots) + (a_2b_1 + a_2b_2 + a_2b_3 + \dots) + \dots \\ &= a_1b_1 + (a_1b_2 + a_2b_1) + (a_1b_3 + a_2b_2 + a_3b_1) + \dots + \\ &\quad (a_1b_n + a_2b_{n-1} + \dots + a_nb_1) + \dots \\ &= \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right).\end{aligned}$$