MATH 4317 – Fall 2012
Week 15 Homework Solutions

Page 166

39. For $n = 1, 2...$ let $I_n = \int_0^{\pi/2} \sin^n x dx$

(a) Show that $\frac{d}{dx}(\cos x \sin^{n-1} x) = (n-1)\sin^{n-2} x - n \sin^n x$

Sol.
Using the chain rule, $\cos^2 x = 1 - \sin^2 x$, and a bit of algebra:

$$\begin{aligned}
\frac{d}{dx}(\cos x \sin^{n-1} x) &= \cos x \frac{d}{dx}(\sin^{n-1} x) + \frac{d}{dx}(\cos x)\sin^{n-1} x \\
&= (n-1)\cos^2 x \sin^{n-2} x - \sin^n x \\
&= \sin^{n-2} x \left(n\cos^2 x - \cos^2 x - \sin^2 x\right) \\
&= \sin^{n-2} x \left(n - n\sin^2 x - 1\right) \\
&= (n-1)\sin^{n-2} x - n\sin^n x
\end{aligned}$$

(b) Show that $I_n = \frac{n-1}{n} I_{n-2}$ if $n \geq 2$

Sol.

$$I_n = \int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \sin x \sin^{n-1} x \, dx$$

Using Integration by Parts

$$\begin{aligned}
I_n = \quad & (-\cos x \sin^{n-1} x)\,\Big|_{x=0}^{\pi/2} & + \int_0^{\pi/2} (n-1)\cos^2 x \sin^{n-2} x \, dx \\
= \quad & 0 & + \int_0^{\pi/2} (n-1)(1 - \sin^2 x)\sin^{n-2} x \, dx
\end{aligned}$$

And now, more algebra:

$$\begin{aligned}
I_n &= (n-1)\left(\int_0^{\pi/2} \sin^{n-2} x \, dx - \int_0^{\pi/2} \sin^n x \, dx\right) \\
&= (n-1)\left(I_{n-2} - I_n\right) \\
&= \frac{n-1}{n} I_{n-2}
\end{aligned}$$

(c) Show that $I_{2n} = \frac{1*3*5...(2n-1)}{2*4*6...(2n)} \frac{\pi}{2}$, $I_{2n+1} = \frac{2*4*6...(2n)}{3*5*6...(2n+1)}$ for $n = 1, 2...$

Sol.
From (b) we know that

$$I_{2n} = \frac{2n-1}{2n} I_{2n-2} = \frac{2n-1}{2n}\frac{2n-3}{2n-2} I_{2n-4} \ldots$$

Continuing this substitution, and recognizing that $I_0 = \pi/2$:

$$I_{2n} = \left(\prod_{k=1}^{n} \frac{2k-1}{2k} \right) I_0 = \frac{\pi}{2} \prod_{k=1}^{n} \frac{2k-1}{2k}$$

With a similar argument, and the knowledge that $I_1 = 1$

$$I_{2n+1} = \frac{2n}{2n+1} I_{2n-1} = \left(\prod_{k=1}^{n} \frac{2k}{2k+1} \right) I_1 = \prod_{k=1}^{n} \frac{2k}{2k+1}$$

(d) Show that $I_0, I_1, I_2 \ldots$ is a decreasing sequence with

$$\lim_{n\to\infty} I_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$$

Sol.
If we define a subset of the real line $X = \{x \in \Re : 0 \le x \le \pi/2\}$ then $\sin X = [0, 1]$ and correspondingly $0 \le \sin^{n+1} x \le \sin^n x \le 1$ for all $x \in X$, so $I_n > I_{n+1} > I_{n+2} \ldots$. With some minimal effort, we can find this family of functions is convergent with $\lim_{n\to\infty} \sin^n x = 0$ for $x \in [0, \pi/2)$ and $\lim_{n\to\infty} \sin^n(\pi/2) = 1$. Using these, one can easily evaluate $\lim_{n\to\infty} I_n = 0$. So the sequence $I_0, I_1 \ldots$ is decreasing and has a limit of 0.

Now we can write

$$1 > \frac{I_{2n+1}}{I_{2n}} > \frac{I_{2n+1}}{I_{2n-1}} = \frac{2n-1}{2n}$$

Letting $n \to \infty$, we can see that

$$\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$$

(e) Show that

$$\lim_{n\to\infty} \frac{2*2*4*4\ldots(2n)*(2n)}{1*3*3*5*5\ldots(2n-1)(2n+1)} = \frac{\pi}{2}$$

Sol.
Here we can recognize the substitution

$$\frac{2*2*4*4\ldots(2n)*(2n)}{1*3*3*5*5\ldots(2n-1)(2n+1)} = \frac{\pi}{2}\frac{I_{2n+1}}{I_{2n}}$$

Now all we must do is evaluate limit using our knowledge from (d)

$$\lim_{n\to\infty} \frac{2*2*4*4\ldots(2n)*(2n)}{1*3*3*5*5\ldots(2n-1)(2n+1)} = \lim_{n\to\infty} \frac{\pi}{2}\frac{I_{2n+1}}{I_{2n}} = \frac{\pi}{2}$$

40(a) Show that if $f : \{x \in \mathbb{R} : x \geq q\} \to \mathbb{R}$ is continuous, then

$$\sum_{i=1}^{n} f(i) = \int_{1}^{n+1} f(x)dx + \sum_{i=1}^{n} \left(f(i) - \int_{i}^{i+1} f(x)dx \right)$$

Sol.

$$\begin{aligned}\sum_{i=1}^{n} f(i) &= \sum_{i=1}^{n} \left(f(i) + \int_{i}^{i+1} f(x)dx - \int_{i}^{i+1} f(x)dx) \right) \\ &= \sum_{i=1}^{n} \int_{i}^{i+1} f(x)dx + \sum_{i=1}^{n} \left(f(i) - \int_{i}^{i+1} f(x)dx \right)\end{aligned}$$

and

$$\sum_{i=1}^{n} \int_{i}^{i+1} f(x)dx = \int_{1}^{2} f(x)dx + \int_{2}^{3} f(x)dx + \dots \int_{n}^{n+1} f(x)dx$$

Since f is a continuous real-valued function on the interval $[i, i+1]$, f is integrable. In addition, by the proposition in p.123,

$$\sum_{i=1}^{n} \int_{i}^{i+1} f(x)dx = \int_{1}^{2} f(x)dx + \int_{2}^{3} f(x)dx + \dots \int_{n}^{n+1} f(x)dx = \int_{1}^{n+1} f(x)dx$$

Therefore,

$$\sum_{i=1}^{n} f(i) = \int_{1}^{n+1} f(x)dx + \sum_{i=1}^{n} \left(f(i) - \int_{i}^{i+1} f(x)dx \right)$$

(b) Show that if $i > 1$ then $\log i - \int_{i}^{i+1} \log x dx$ differs from $-1/2i$ by less than $1/6i^2$.

Sol.

$$\begin{aligned}\log i - \int_{i}^{i+1} \log x dx &= \int_{i}^{i+1} \log i - \log x dx \\ &= \int_{i}^{i+1} \log \left(\frac{i}{x} \right) dx \qquad (1)\end{aligned}$$

We are going to use change of variables

$$\int_{\varphi(a)}^{\varphi(b)} f(x)dx = \int_{a}^{b} f(\varphi(t))\varphi'(t)dt$$

where

$$\begin{aligned}\varphi(x) &= i + x \\ f(x) &= \log \left(\frac{i}{x} \right) \\ f(\varphi(t)) &= \log \left(\frac{i}{i+t} \right)\end{aligned}$$

Hence the integral (1) becomes

$$\int_i^{i+1} \log\left(\frac{i}{x}\right) dx = \int_0^1 \log\left(\frac{i}{i+t}\right) dt = -\int_0^1 \log\left(\frac{i+t}{i}\right) dt = -\int_0^1 \log\left(1+\frac{t}{i}\right) dt$$

The Taylor series for $\log(1+x)$ at the point 0 is

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

Therefore,

$$\begin{aligned}
-\int_0^1 \log\left(1+\frac{t}{i}\right) dt &= -\int_0^1 \left[\frac{t}{i} - \frac{1}{2}\left(\frac{t}{i}\right)^2 + \dots\right] dt \\
&= -\frac{1}{2i}[t^2]_0^1 + \frac{1}{6i^2}[t^3]_0^1 - \dots \\
&= -\frac{1}{1*2*i} + \frac{1}{2*3*i^2} - \dots
\end{aligned}$$

We can take the sums starting from the second term:

$$\begin{aligned}
\frac{1}{(2*3)i^2} - \frac{1}{(3*4)i^3} &= \frac{3*4*i^3 - 2*3*i^2}{2*3*3*4*i^5} \\
\frac{1}{(2*3)i^2} - \frac{1}{(3*4)i^3} + \frac{1}{(4*5)i^4} &= \frac{3*4*4*5*i^7 - 2*3*4*5*i^6 + 2*3*3*4*i^5}{2*3*3*4*4*5*i^9} \\
&\vdots
\end{aligned}$$

And recognize that for $i \geq 1$, the highest-order term in the numerator dominates and, in fact, the numerator is bounded between zero and the value of that highest-order term. Now, dropping the remaining numerator terms, we have

$$0 < \frac{1}{6i^2} - \frac{1}{20i^3} + \frac{1}{30i^4} \dots < \frac{1}{6i^2}$$

Thus,

$$-\frac{1}{2i} < \log i - \int_i^{i+1} \log x dx < -\frac{1}{2i} + \frac{1}{6i^2}$$

(c) Use part (a) with $f = \log$, part (b), and Prob. 22, Chap. VI to prove

$$\lim_{n\to\infty} \left(\log n! - \left(n + \frac{1}{2} \right) \log n + n \right)$$

exists.

Sol.

$$\begin{aligned}
\log n! - \left(n + \frac{1}{2} \right) \log n + n &= \sum_{i=1}^{n} \log i - \left(n + \frac{1}{2} \right) \log n + n \\
&= \int_1^{n+1} \log x dx + \sum_{i=1}^{n} \left(\log i - \int_i^{i+1} \log x dx \right) - \left(n + \frac{1}{2} \right) \log n + n \\
&= (n+1)\log(n+1) - \left(n + \frac{1}{2} \right) \log n + \sum_{i=1}^{n} \left(\log i - \int_i^{i+1} \log x dx \right) \\
&= n \log \left(\frac{n+1}{n} \right) + \log \left(\frac{n+1}{n} \right) + \frac{1}{2} \log n + \sum_{i=1}^{n} \left(\log i - \int_i^{i+1} \log x dx \right)
\end{aligned}$$

Here we have used part (a). For the first term and the second term,

$$\lim_{n\to\infty} n \log \left(\frac{n+1}{n} \right) = \lim_{n\to\infty} \log \left(1 + \frac{1}{n} \right)^n = \log e = 1$$

$$\lim_{n\to\infty} \log \left(\frac{n+1}{n} \right) = \lim_{n\to\infty} \log \left(1 + \frac{1}{n} \right) = 0$$

For the third and fourth term,

$$\begin{aligned}
\frac{1}{2} \log n + \sum_{i=1}^{n} \left(\log i - \int_i^{i+1} \log x dx \right) &< \frac{1}{2} \log n + \sum_{i=1}^{n} -\frac{1}{2i} + \frac{1}{6i^2} \\
&< -\frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right) + \sum_{i=1}^{n} \frac{1}{6i^2}
\end{aligned}$$

We have used part (b). From Prob. 22, Chap. VI, we know that

$$\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right)$$

is positive, decreases as n increases and hence that the sequence of these numbers converges to a limit between 0 and 1. Furthermore, the summation

$$\sum_{i=1}^{n} \frac{1}{6i^2}$$

converges as n goes to ∞ because $i > 1$. Therefore, we have the existence of the limit

$$\lim_{n\to\infty} \left(\log n! - \left(n + \frac{1}{2} \right) \log n + n \right)$$

(d) Use part (e) of the preceding problem to compute the above limit, thus obtaining

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1$$

Sol. We know that the sequence in (c) converges.

$$\lim_{n\to\infty} \log n! - \left(n + \frac{1}{2}\right)\log n + n = C$$

where C is a constant. Taking exponential on both sides gives

$$\begin{aligned}
\exp\left(\log n! - \left(n+\frac{1}{2}\right)\log n + n\right) &= \exp\left(\log n! - \log n^{(n+1/2)} + n\right) \\
&= \frac{n!}{n^{(n+1/2)}e^{-n}} \\
&= \frac{n!}{n^n e^{-n}\sqrt{n}} = e^C
\end{aligned}$$

Hence we can get

$$n! = n^n e^{-n}\sqrt{n}e^C \tag{2}$$

We are going to show that e^C is $\sqrt{2\pi}$ by using Wallis' product and several approximation so that

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1$$

The Wallis' product is

$$\lim_{n\to\infty} \frac{2\cdot 2\cdot 4\cdot 4\cdot \cdots \cdot (2n)\cdot(2n)}{1\cdot 1\cdot 3\cdot 3\cdot \cdots \cdot (2n-1)\cdot(2n-1)\cdot(2n+1)} = \frac{\pi}{2}$$

Taking square roots on both sides yields

$$\lim_{n\to\infty} \frac{2\cdot 4\cdot \ldots (2n)}{1\cdot 3\cdot \cdots \cdot (2n-1)\cdot\sqrt{(2n+1)}} = \sqrt{\frac{\pi}{2}}$$

Multiplying $2\cdot 4\cdot \ldots (2n)$ at numerator and denominator gives

$$\lim_{n\to\infty} \frac{2^2\cdot 4^2\cdot \ldots (2n)^2}{1\cdot 2\cdot 3\cdot 4\cdot \cdots \cdot (2n-1)\cdot 2n\cdot\sqrt{(2n+1)}} = \sqrt{\frac{\pi}{2}}$$

We rewrite the above equation as

$$\lim_{n\to\infty} \frac{(2^n n!)^2}{(2n)!\cdot\sqrt{(2n+1)}} = \sqrt{\frac{\pi}{2}}$$

Now we are going to do approximation, i.e,

$$\frac{(2^n n!)^2}{(2n)!\cdot\sqrt{(2n)}} \simeq \sqrt{\frac{\pi}{2}}$$

Substituting the equation (2) into the above equation gives

$$
\begin{aligned}
\frac{2^{2n}(n!)^2}{(2n)!\sqrt{(2n)}} &= \frac{2^{2n}(n^n e^{-n}\sqrt{n}e^C)^2}{(2n)^{2n}e^{-2n}\sqrt{2n}e^C\sqrt{(2n)}} \\
&= \frac{2^{2n}n^{2n}e^{-2n}ne^{2C}}{(2n)^{2n}e^{-2n}\sqrt{2n}e^C\sqrt{(2n)}} \\
&= \frac{e^C}{2} = \sqrt{\frac{\pi}{2}}
\end{aligned}
$$

Note that

$$(2n)! = (2n)^{(2n)}e^{-(2n)}\sqrt{(2n)}e^C$$

Therefore,

$$e^C = \sqrt{2\pi}$$

Hence

$$\lim_{n\to\infty}\frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1$$