Group 1
Math 4317 Homework 2

2a
(b-a)-(-(a-b))=(b-a)+(a-b)=(b+(-a))+(a+(-b))=b+(-a)+a+(-b)=b+(-b)=0. Therefore, because x-y=0, for x=(b-a) and y=-(a-b), b-a=-(a-b).

2b
Let a, b, c, d∈**R**. Because **R** is closed for addition and additive inverses, here exists some f∈**R** such that f=a-b. Then, (a-b)(c-d)=f(c-d)=fc-fd=(a-b)c-(a-b)d=(ac-bc)-(ad-bd). By **1a**, this=(ac-bc)+(bd-ad)=ac-bc+bd-ad=ac+bd-ad-bc=(ab+bd)+(-1)(ad+bc)=(ab+bd)-(ad+bc). Therefore, (a-b)(c-d)=(ac+bd)-(ad+bc)

4a
No. 223*7=1561<1562=22*71, meaning 223/71<22/7

4b
No. 265*780=206700<206703=1351*153, meaning 265/153<1351/780

6
Let a,b,x,y∈**R** such that a<x<b and a<y<b. Then, -b<-y<-a, and a+(-a)<x+(-a)<b+(-a). Because -y<-a, it is also true that a+(-y)<x+(-y)<x+(-a)<b-a, meaning , a+(-y)<x+(-y)<b-a. Similarly, because -b<-y, a+(-b)<a+(-y)<x+(-y)<b-a, meaning a+(-b)<x+(-y)<b-a and a-b<x-y<b-a, or -(b-a)<x-y<b-a, or |x-y|<b-a. Therefore, |y-x|<b-a.

7a
For some a,b∈**R**, if a≤b, a-b≤0, meaning |a-b|=-(a-b)=b-a, and $\frac{a+b+|a-b|}{2}=\frac{a+b+b-a}{2}=\frac{b+b}{2}=\frac{2b}{2}=b=\max(a,b)$. If a>b, a-b>0 and |a-b|=a-b, meaning $\frac{a+b+|a-b|}{2}=\frac{a+b+a-b}{2}=\frac{a+a}{2}=\frac{2a}{2}=a=\max(a,b)$. Because, for all r,s∈**R** s<r, s=r, or s>r, $\frac{a+b+|a-b|}{2}$= $\max(a,b)$ for all a,b∈**R**

7b
Let a,b∈**R**. Because a,b are from a well-ordered domain, a<b, a=b, or a>b. If a<b, -b>-a, meaning -max(-a,-b)=-(-b)=b=min(a,b). If a=b, -max(-a,-b)=-max(-a,-a)=-(-a)=a=min(a,b). If a>b, -a<-b, and -max(-a,-b)=-(-a)=a=min(a,b). Therefore, for any a,b∈**R**, min(a,b)=-max(-a,-b).

9
The empty set is bounded both from above and from below, because for any r∈**R**, r>x for all x∈∅ and r<x for all x∈∅, although it does not have a least upper bound or greatest lower bound because **R** has neither.

10a
L.U.B=1, because each a≠1 in the set is less is less than 1, while any real number a lower than 1 would not include the element 1 within the bound
G.L.B=0, because the elements of the set are approaching 0, meaning for any selected element a>0, an item b could be found in the sequence such that b<a.

10b
L.U.B=1/2, because the elements of the set are approaching 1/2, meaning for any selected element a<1/2, an item b could be found in the sequence such that b>a.

Group 1
Math 4317 Homework 2

G.L.B=1/3, because each a≠1/3 in the set is less than 1/3, while any real b greater than 1/3 would not include the element 1/3 within the bound

10c
Let f(n)=$\sqrt{2+f(n-1)}$ for all n∈**Z**>0 and f(0)=$\sqrt{2}$.
L.U.B=2, because the limit of the sequence is 2, as shown:

$X_{i+1}=\sqrt{2+X_i}$
$(X_{i+1})^2=2+X_i$
For sufficiently large i, $X_{i+1}=X_i$
$X_i^2-X_i-2=0$
$(X_i-2)(X_i+1)=0$
Therefore, $\lim_{n\to\infty} X_i$ = {-1,2}, and since -1<$X_0=\sqrt{2}$, $\lim_{n\to\infty} X_i = 2$.

G.L.B=$\sqrt{2}$, because each a≠$\sqrt{2}$ in the set is less than $\sqrt{2}$, while any real b greater than $\sqrt{2}$ would not include the element $\sqrt{2}$ within the bound

11
Let a>1∈**R** and S={a, a^2, a^3, …}. For all n>3∈**Z**, a^{n+2}-a^{n+1}=a(a^{n+1}-a+n) which means, because a>1, the difference between each successive a^n and a^{n+1} must be increasing. Then, there exists some n∈$\mathbf{Z}^+$ such that 1/n<a-1. Let T={x+1/n|x∈T} be a set with a∈T where t_i=i/n. **N**⊆T, because each element of Z can be written as k*1/n, where k∈**Z** and, therefore, the set T must not have an upper bound because **N** does not. As a result, because $t_i \le a^i$ for all i, S must also be unbounded from above.

12
Let X be some set such that X≠{x∈**R**|x<a} and X≠{x∈**R**|x≤a} for any a∈**R**. Then, there are 2 cases. Either the elements of X have a least upper bound, or they do not. If they do have an upper bound, then let this bound be a. Then, there must exist some b<a such that b∉X. Then, because X∪Y=**R**, b∈Y, meaning that an element of Y is not greater than all elements of X, which is a contradiction. If X does not have an upper bound, then it must contain all elements of **R** because it is unbounded above and, because all elements of X are greater than those of Y and X∪Y=**R**, it must also be unbounded below. This is a contradiction, because Y must then be empty, or it would contain an element equal to one in X. In either case, a contradiction occurs, meaning that there must exist some a∈**R** such that X≠{x∈**R**|x<a} or X≠{x∈**R**|x≤a}.

13
Let S_1, S_2⊆**R** be non-empty with least upper bounds of a and b, let T={a+b|a∈S_1, b∈S_2}. Then, for all x∈S_1, x≤a, and for all y∈S_2, y≤b, meaning all x+y≤a+b and a+b is a upper bound of T. Now, let some m be an upper bound of T such that m<a+b. Then, (a+b)-m=e, for some e>0∈**R**. Because a is the least upper bound of S_1, there must exist some s_1∈S_1 such that a-s_1<e/2. Otherwise, a-e/2 would be the least upper bound of S_1. Similarly, there must exist some s_2∈S_2 such that a-s_2<e/2. Therefore, because s_1+s_2>a+b-e and s_1+s_2∈T, s_1+s_2>m, meaning m cannot be a maximum which is a contradiction. Therefore, a+b must be the least upper bound of T.

16
For any base b, b-nary expansions, let $a_0.a_1a_2$… be any real number, where a_0 is any integer and a_i such that i >0 and a_i∈[0,b)⊆**Z**, meaning $a_0.a_1a_2$… = a_0 + a_1/b^1 + a_2/b^2 + …. Then, as with decimal expansions of real numbers, the set {$a_0.a_1…a_n$|n∈**N**} is non-empty and bounded from above, so it has a least upper

bound, which is represented by the infinite b-nary expansion. Additionally, for m, n∈**Z**$^+$, $a_0.a_1...a_m \le a_0.a_1...a_n = a_0.a_1...a_m + a_{m+1}b^{-m-1} + ... + a_nb^{-n} \le a_0.a_1...a_m + (b-1)*b^{-m-1} + ... + (b-1)*b^{-n} < a_0.a_1...a_m + b^{-m}$, resulting in $a_0.a_1...a_m \le a_0.a_1... < a_0.a_1...a_m + b^{-m}$. Together, these show that any real number x can be represented by an infinity b-nary expansion. For this, apply the fact that, for any N∈**Z**$^+$, there exists n∈**Z** such that $n/N \le x < (n+1)/N$, where $N=b^m$ for some m∈**Z**$^+$. The result of this can be re-written as $a_0.a_1...a_m \le x < a_0.a_1...a_m + 10^{-m}$. As m increases to larger integer values, the b-nary expansion gets closer to the actual value of x. Therefore, b-nary representations of real numbers have properties analogous to those decimal numbers possess.