Problem 10 c)

$f_{n+1} = \sqrt{2+f_n}$, $f_1 = \sqrt{2}$

a) $f_{n+1} \geq f_n$, i.e., sequence is monotone increasing.

Proof: $f_1 = \sqrt{2} < \sqrt{2+\sqrt{2}} = f_2$ ✓

Induction step: Assume $f_n \geq f_{n-1}$.
To show $f_{n+1} \geq f_n$.

$f_{n+1} = \sqrt{2+f_n} \geq \sqrt{2+f_{n-1}} = f_n$.

b) $f_n \leq 2$ $n = 1, 2, \ldots$

Proof: $f_1 = \sqrt{2} < 2$ ✓ Induction step.

assume $f_n \leq 2$. To show $f_{n+1} \leq 2$
$f_{n+1} = \sqrt{2+f_n} \leq \sqrt{2+2} = \sqrt{4} = 2$.

c) f_n monotone increasing sequence and bounded. Hence it converges to some limit a. We have

$a = \sqrt{2+a}$, i.e., $a = 2$.

l.u.b $= 2$, g.l.b $= \sqrt{2}$

Problem 11

$a \in \mathbb{R}$, $a > 1$. By L.U.B 2 in the book there ex a positive integer n so that $\frac{1}{n} < a-1$ and hence $a > 1 + \frac{1}{n}$.

Now $(1+\frac{1}{n})^n = \sum_{k=0}^{n} \binom{n}{k} (\frac{1}{n})^k = 1 + n \cdot \frac{1}{n} + \binom{n}{2} \frac{1}{n^2} + \cdots$

by the binomial formula. This number is clearly > 2. Hence $a^n > 2$.

The set $\{2, 2^2, 2^3, \ldots\}$ is not bounded. Pick any positive integer N. There ex k pos. integer so that $2^k > N$. Hence $a^{nk} \geq 2^k > N$ and $\{a, a^2, \ldots\}$ is not bounded.

Problem 12

Every element of X is less than every element of Y. Thus X is bounded above. Since it is not empty there ex. a l.u.b. a for X. Hence

$$X \subset \{x \in \mathbb{R} : x \leq a\}.$$

Let $z \in \{x \in \mathbb{R} : x < a\}$ but $z \notin X$. Then $z \in Y$ because $X \cup Y = \mathbb{R}$. This cannot be since z would have to be an upper bound for X.
Hence

$$\{x \in \mathbb{R} : x < a\} \subset X$$

Since $\{x \in \mathbb{R} : x \leq a\} = \{x \in \mathbb{R} : x < a\} \cup \{a\}$

$X = \{x \in \mathbb{R} : x \leq a\}$ or $X = \{x \in \mathbb{R} : x < a\}$.