Analysis I HW 3

Mingtao Xu 902853305 & Hamid Mohammadi 902870256

1 Verify that the following are metric spaces:

a) all n-tuples of real numbers, with

$$d((x_1,\dots,x_n),(y_1,\dots,y_n)) = \sum_{i=1}^{n} |x_i - y_i| .$$

→ ① $\sum_{i=1}^{n} |x_i - y_i| = |x_1-y_1| + \dots + |x_n - y_n| \geq 0$ ✓

→ ② $d((x_1,\dots,x_n),(y_1,\dots,y_n)) = 0 \Rightarrow \sum_{i=1}^{n} |x_i - y_i| = 0 \Rightarrow$

$\forall i \in \mathbb{N} \quad x_i = y_i \Rightarrow (x_1,\dots,x_n) = (y_1,\dots,y_n) \quad (\Rightarrow)$ ✓

$(x_1,\dots,x_n) = (y_1,\dots,y_n) \Rightarrow |x_1 - y_1| = 0, \dots, |x_n - y_n| = 0 \Rightarrow$

$\sum_{i=1}^{n} |x_i - y_i| = 0 \Rightarrow d((x_1,\dots,x_n),(y_1,\dots,y_n)) = 0 \quad (\Leftarrow)$ ✓

→ ③ $d((x_1,\dots,x_n),(y_1,\dots,y_n)) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |y_i - x_i| =$

$d((y_1,\dots,y_n),(x_1,\dots,x_n))$ ✓

→ ④ $d((x_1,\dots,x_n),(y_1,\dots,y_n)) + d((y_1,\dots,y_n),(z_1,\dots,z_n)) =$

$\sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i| = \sum_{i=1}^{n} (|x_i - y_i| + |y_i - z_i|) \geq$

$\sum_{i=1}^{n} (|x_i - y_i + y_i - z_i|) = \sum_{i=1}^{n} |x_i - z_i| = d((x_1,\dots,x_n),(z_1,\dots,z_n))$ ✓

b) all bounded infinite sequences $x=(x_1, x_2, \dots)$ of elements of $\mathbb{R}$, with:

$$d(x,y) = \text{l.u.b.}\{|x_1-y_1|, |x_2-y_2|, \dots\}$$

first 3 axioms are very trivial. need to show the triangular inequality!

$$d(x,y) + d(y,z) = \text{L.u.b.}\{|x_1-y_1|, |x_2-y_2|, \dots\} +$$

$$\text{L.u.b.}\{|y_1-z_1|, |y_2-z_2|, \dots\} \overset{\text{Problem 13}}{\underset{\text{Chapter II}}{=}}$$

$$\text{L.u.b.}\{(|x_1-y_1|+|y_1-z_1|), (|x_2-y_2|+|y_2-z_2|), \dots\} \geqslant$$

$$\text{L.u.b.}\{(|x_1-y_1+y_1-z_1|), (|x_2-y_2+y_2-z_2|), \dots\} =$$

$$\text{L.u.b.}\{|x_1-z_1|, |x_2-z_2|, \dots\} = d(x,z)$$

C) $(E_1 \times E_2, d)$ where $(E_1, d_1), (E_2, d_2)$ are metric spaces and d is given by $d((x_1,x_2),(y_1,y_2)) = \max\{d_1(x_1,y_1), d_2(x_2,y_2)\}$.

Again we only show that triangula inequality holds since other 3 axioms are very trivial.

$$d((x_1,x_2),(y_1,y_2)) + d((y_1,y_2),(z_1,z_2)) \geq d((x_1,x_2),(z_1,z_2))$$

$$LHS = \max\{\underbrace{d_1(x_1,y_1)}_{a}, \underbrace{d_2(x_2,y_2)}_{b}\} + \max\{\underbrace{d_1(y_1,z_1)}_{c}, \underbrace{d_2(y_2,z_2)}_{d}\}$$

$$= \frac{a+b+|a-b|}{2} + \frac{c+d+|c-d|}{2} = \frac{1}{2}(a+b+c+d+|a-b|+|c-d|)$$

$$\geq \frac{1}{2}(a+b+c+d+|a-b+c-d|) =$$

$$\frac{1}{2}(d_1(x_1,y_1)+d_2(x_2,y_2)+d_1(y_1,z_1)+d_2(y_2,z_2)+|d_1(x_1,y_1)+d_1(y_1,z_1) \dots|)$$

Continue...

$$\geq \frac{1}{2}\left(d_1(x_1, z_1) + d_2(x_2, z_2) + |d_1(x_1, z_1) - d_2(x_2, z_2)| \right)$$

$$= \max\{d_1(x_1, z_1), d_2(x_2, z_2)\} = d((x_1, x_2), (z_1, z_2))$$

6 Show that the subset of E^2 given by $\{(x_1, x_2) \in E^2 : x_1 x_2 = 1, x_1 > 0\}$ is closed.

Let (x_{1i}, x_{2i}) be any sequence in S that converges in E^2.

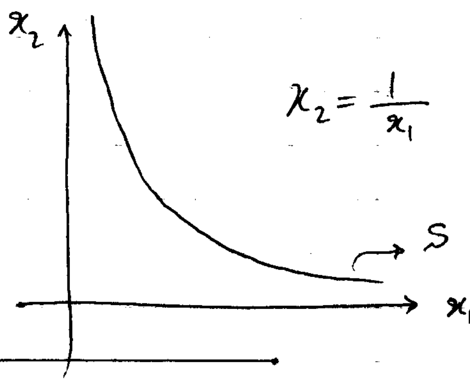
$$x_{2i} = \frac{1}{x_{1i}} \quad \text{and} \quad x_{1i} \xrightarrow[i\to\infty]{} x_1$$

If $x_1 = 0$ then $\lim x_{2i} = \lim \frac{1}{x_{1i}}$ doesn't exist

and hence $x_1 > 0$. $\quad x_{2i} \xrightarrow[i\to\infty]{} x_2$

$$x_{2i} - x_2 = \frac{1}{x_{1i}} - x_2 = \frac{1 - x_{1i} x_2}{x_{1i}}$$

So $1 - x_{1i} x_2 \to 0$ as $i \to \infty \Rightarrow x_1 x_2 = 1$



7 Proof:

Suppose $b = $ g.l.b. S and suppose $b \notin S$, then $b \in \complement S$ and $\complement S$ is open since S is closed. therefore there exist $B_\varepsilon(b)$ in $\complement S$ and since $S \subset R$ all the elements in S are greater than $(b+\varepsilon)$ which make the $(b+\varepsilon) = $ g.l.b. S ↯ contradiction since $b = $ g.l.b. S therefore $b \in S$.

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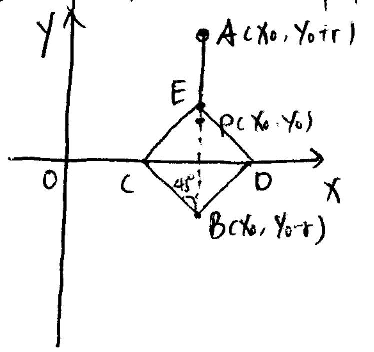
2. Solution: Denote the center as $P(x_0, y_0)$ and radius as r $(r>0)$, then the open ball, by definition, is given by ~~{(x, y)∈~~
$\{(x, y) \in E^2,\ d((x_0, y_0), (x, y)) < r$.

i). If $y_0 > 0$ and $|r| > |y_0|$, $(r > y_0 > 0)$.



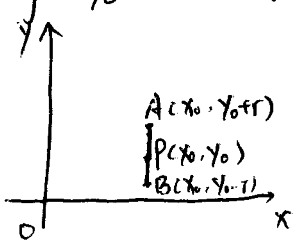
The points that satisfy $d((x_0,y_0),(x,y)) = r$ are the points along EDBC (except point E), and point A.

So the open ball is the set of points inside EDBC (boundry excluded) and points along EA line section (E included but A excluded).

And EDBC is a square with ~~ED~~ CD the diagonal line.

Coordinates:
$A(x_0, y_0+r)$, $B(x_0, y_0-r)$
$C(x_0-r+y_0, 0)$, $D(x_0+r-y_0, 0)$
$E(x_0, r-y_0)$.

ii) If $y_0 > 0$ and $|y_0| \geq |r|$, $(y_0 \geq r > 0)$.



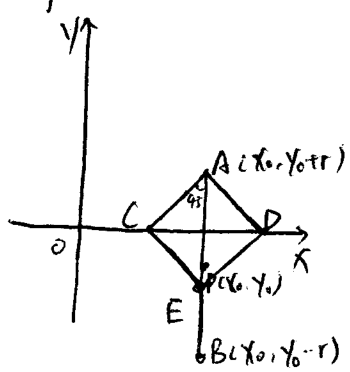
In this case. $\forall\ x \neq x_0$.

$$\left.\begin{array}{l} d((x_0,y_0),(x,y)) = |y_0| + |y| + |x - x_0| \\ \text{And } |y_0| > r \\ \quad |x - x_0| > 0 \end{array}\right\}$$

$\Rightarrow d((x_0,y_0),(x,y)) > r + 0 = r$.

So the open ball is the set of points on line section AB, with point A and point B excluded, since $d(A, P) = d(B, P) = r$.

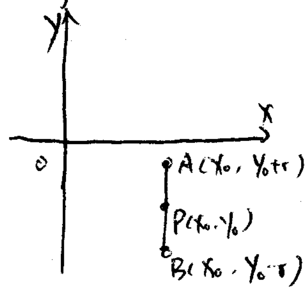
iii). If $y_0 < 0$ and $|r| > |y_0|$, $(r > -y_0 > 0 > y_0)$.



The open ball is the set of points inside square ADEC (boundry excluded) and points along EB line section (E included but B excluded). Similar as i)

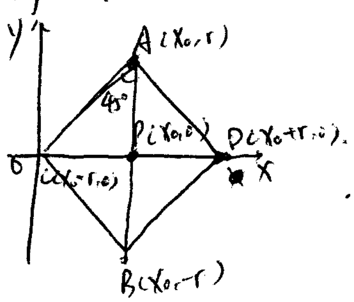
Coordinates: $A(x_0, y_0+r)$; $B(x_0, y_0-r)$
$C(x_0-r-y_0, 0)$; $D(x_0+r+y_0, 0)$.
$E(x_0, -r-y_0)$.

iv). If $y_0 < 0$ and $|r| \le |y_0|$ $\quad -y_0 \ge r > 0 > y_0$



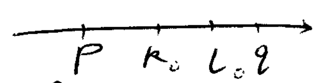
Similar as ii). in this case. the open ball is the set of points along line section AB with A and B excluded.

v). If $y_0 = 0$,

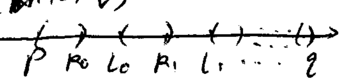


The open ball is the set of points inside square ADBC (boundary excluded). All the points on ADBC has $d((x_0, y_0), (x, y)) = r$

5. Solution: Let S be a bounded open subset of R with G.L.B p and L.U.B q. so $S \subseteq (p, q)$.
Since S is open, $\forall x \in S$, $\exists r > 0$, s.t. $B_r(x) \subset S$.
In this way. ~~for~~ $\forall k_0, l_0$ that $p < k < l < q$ and ~~$[k_0, l_0] \notin S$~~ the points between k_0, l_0 are not in S, there must be $[k_0, l_0] \notin S$. which means $[k_0, l_0]$ is closed. Then $S = (p, k_0) \cup (l_0, q)$.



Do the same to all the k_i, l_i s.
And $S = (p, k_0) \cup (l_0, k_1) \cup \cdots \cup (l_n, k_{n+1}) \cup (l_{n+2}, q)$
$= (p, k_0) \cup [\bigcup_{i=0}^{n} (l_i, k_{i+1})] \cup (l_{n+2}, q)$.



10. Solution: $\lim_{n\to\infty} P_n = P$ ~~as~~. Define $S = \{P, P_1, P_2, \cdots\}$.
Then $P \in S$.
Assume S is not closed, then $\complement S$ is not open.
So $\forall$ arbitrary small $\varepsilon > 0$, $\exists$ $p' \in \complement S$ such that $B_\varepsilon(p') \cap S \ne \phi$.
which means $\complement S$ do not contain an open ball of center p'.
In this way. $\lim_{n\to\infty} P_n = p' \in \complement S$, but $\lim_{n\to\infty} = P$
So $p' = P$, but $P \in S$ and $p' \in \complement S$. This is a contradiction.
And S must be closed.

11. Solution: Since $a_1, a_2, \cdots$ converges to a. so pick an arbitrary small ε, we can find a N such that $\forall i > N$.
$|a_i - a| < \varepsilon$.

Proposition on P48.

Then $\lim_{n\to\infty} \left(\frac{\sum_{i=1}^{n} a_i}{n} \right) = \lim_{n\to\infty} \left(\frac{\sum_{i=1}^{N} a_i}{n} \right) + \lim_{n\to\infty} \left(\frac{\sum_{i=N+1}^{n} a_i}{n} \right)$

N is a fixed number, so $\lim_{n\to\infty} \left(\frac{\sum_{i=1}^{N} a_i}{n} \right) = 0$. ~~$\lim_{n\to\infty} \left(\frac{\sum_{i=1}^{n} a_i}{n}\right) = \lim_{n\to\infty} \left(\frac{\sum_{i=N+1}^{n} a_i}{n}\right)$~~

$$\forall i > N. \quad |a_i - a| < \varepsilon \iff -\varepsilon < a_i - a < \varepsilon$$
$$\iff -\frac{\varepsilon}{n} < \frac{a_i - a}{n} < \frac{\varepsilon}{n}$$
$$\iff -\frac{\varepsilon}{n} \cdot (n - N) < \sum_{i=N+1}^{n} \frac{a_i - a}{n} < +\frac{\varepsilon}{n} \cdot (n - N).$$
$$\iff -\varepsilon + \frac{N}{n}\varepsilon < \sum_{i=N+1}^{n} \frac{a_i - a}{n} < \varepsilon - \frac{N}{n}\varepsilon$$
$$\lim_{n\to\infty} \left(-\varepsilon + \frac{N}{n}\varepsilon\right) < \lim_{n\to\infty} \sum_{i=N+1}^{n} \frac{a_i - a}{n} < \lim_{n\to\infty} \left(\varepsilon - \frac{N}{n}\varepsilon\right)$$
$$\iff -\varepsilon < \lim_{n\to\infty} \sum_{i=N+1}^{n} \frac{a_i}{n} - a < \cancel{\lim_{n}}\ \varepsilon.$$

As ε can be any number greater than zero.

So $\lim_{n\to\infty} \sum_{i=N+1}^{n} \frac{a_i}{n} - a = 0 \iff \lim_{n\to\infty} \sum_{i=1}^{n} \frac{a_i}{n} = a$