12. Note: for $a > b \geq 2$ $\quad a + 1/a^2 > b + 1/b^2$

$a > b \Rightarrow a - b > 0$

$1/a^2 - 1/b^2 = b^2 - a^2/b^2a^2 = (b-a)(b+a)/b^2a^2 = -(a-b)(b+a)/b^2a^2$

$(a + 1/a^2) - (b + 1/b^2) > 0$

$(a-b) + (1/a^2 - 1/b^2) > 0$

$(a-b) + (-(a-b)(b+a)/b^2a^2) > 0$

$(a-b)(1 - b+a/b^2a^2) > 0$

$(a-b)(1 - 1/ba^2 - 1/b^2a) > 0$

since $a > b \geq 2 \Rightarrow ba^2 \geq 8 \quad b^2a \geq 8$

$(a-b)(1 - 1/ba^2 - 1/b^2a) > (a-b)(1 - (\frac{1}{8} + \frac{1}{8})) > 0$

$\Rightarrow (a + 1/a^2) - (b + 1/b^2) > (a-b)(3/4)$

$\therefore$ x_n is monotonicaly increasing.

Suppose X_n is bounded

$\Rightarrow X_n$ converges to some ~~point~~ limit p.

note: for $n \geq 2$, $x_n \geq 2$.

$\exists X_{N(\varepsilon)} > p - \varepsilon \quad \forall \varepsilon > 0$

Let ε be $1/p^2$

$X_{N(\varepsilon)+1} = X_{N(\varepsilon)} + \frac{1}{X_{N(\varepsilon)}^2} > p - \frac{1}{p^2} + \frac{1}{(p - \frac{1}{p^2})^2}$

since $p \geq 2$ $\quad p - 1/p^2 > 0$

$(p - 1/p^2)^2 < p^2 \Rightarrow \frac{1}{(p - 1/p^2)^2} > \frac{1}{p^2}$

$X_{N(\varepsilon)+1} > p - \frac{1}{p^2} + \frac{1}{(p - 1/p^2)^2} > p - 1/p^2 + 1/p^2 = p$

$X_{N(\varepsilon)+1} > p$ which is a contradiction $X_{N(\varepsilon)+1}$ is supposed to be less than p. X_n is monotonically increasing.

$\therefore$ X_n is not bounded.

13) $x - \dfrac{1}{2+\frac{1}{2+x}}$ <0 when $0<x<\sqrt{2}-1$
>0 when $x>\sqrt{2}-1$

$$x - \frac{1}{\frac{2(2+x)+1}{2+x}} = x - \frac{2+x}{4+2x+1} = x - \frac{x+2}{2x+5} = \frac{2x^2+5x-x-2}{2x+5} = \frac{2x^2+4x-2}{2x+5}$$

$$\frac{2x^2+4x-2}{2x+5} < 0 \Rightarrow x^2+2x-1<0$$

$$x < \frac{-2\pm\sqrt{4+4}}{2}$$

$x < \dfrac{-2\pm 2\sqrt{2}}{2} = -1\pm\sqrt{2}$. we are only looking at $x>0$, so
$\boxed{x<\sqrt{2}-1}$

$$\frac{1}{2+\frac{1}{2+x}} - x = \frac{x+2}{2x+5} - x = \frac{x+2-(2x^2+5x)}{2x+5} = \frac{x+2-2x^2-5x}{2x+5}$$

$$= \frac{-2x^2-4x+2}{2x+5} > 0 \Rightarrow 2x^2+4x-2>0 \Rightarrow x^2+2x-1>0$$

$$x > \frac{-2\pm 2\sqrt{2}}{2} \Rightarrow \boxed{x>\sqrt{2}-1}$$

Let the first term in the sequence be a_1 and let every subsequent terms a_n, with $n=2,3,4,\ldots$
Note that $a_{n+2} = \dfrac{1}{2+\frac{1}{2+a_n}}$.

Hence, we have shown above that if $a_n<\sqrt{2}-1$, then $a_n<a_{n+2}$.
Similarly, if $a_n>\sqrt{2}-1$, then $a_n>a_{n+2}$ (assuming all $a_n>0$, which will be proven later).
We will now prove that if $a_n<\sqrt{2}-1$, then $a_{n+2}<\sqrt{2}-1$ and
if $a_n>\sqrt{2}-1$, then $a_{n+2}>\sqrt{2}-1$, by
contradiction.

13 cont) Assume $a_n < \sqrt{2} - 1$ and $a_{n+2} \geq \sqrt{2} - 1$

$$a_{n+2} = \frac{1}{2 + \frac{1}{2+a_n}} \geq \sqrt{2} - 1$$

$$\frac{a_n + 2}{2a_n + 5} \geq \sqrt{2} - 1$$

$$a_n + 2 \geq (\sqrt{2} - 1)(2a_n + 5)$$

$$a_n + 2 \geq 2\sqrt{2}a_n - 2a_n + 5\sqrt{2} - 5$$

$$a_n - 2\sqrt{2}a_n + 2a_n \geq 5\sqrt{2} - 7$$

$$a_n(3 - 2\sqrt{2}) \geq 5\sqrt{2} - 7$$

$$a_n \geq \frac{5\sqrt{2} - 7}{3 - 2\sqrt{2}}$$

$$a_n \geq \frac{5\sqrt{2} - 7}{3 - 2\sqrt{2}} \cdot \frac{3 + 2\sqrt{2}}{3 + 2\sqrt{2}}$$

$$a_n \geq \frac{15\sqrt{2} - 21 + 20 - 14\sqrt{2}}{9 - 8}$$

$$a_n \geq \sqrt{2} - 1$$

But by assumption, $a_n < \sqrt{2} - 1$, so this is a contradiction.

Assume $a_n > \sqrt{2} - 1$ and $a_{n+2} \leq \sqrt{2} - 1$

$$\frac{a_n + 2}{2a_n + 5} \leq \sqrt{2} - 1$$

$$a_n \leq \frac{5\sqrt{2} - 7}{3 - 2\sqrt{2}}$$

$$a_n \leq \sqrt{2} - 1$$

But by assumption, $a_n > \sqrt{2} - 1$, so this is a contradiction. ∎

Since $a_1 = \frac{1}{2} > \sqrt{2} - 1$, then $a_1, a_3, \ldots$ is monotonically decreasing and bounded by $\sqrt{2} - 1$. Similarly, $a_2 = \frac{2}{5} < \sqrt{2} - 1$, so $a_2, a_4, \ldots$ is monotonically increasing and bounded by $\sqrt{2} - 1$.

13 cont) We have just shown that all elements $a_1, a_3, a_5, \ldots$ are greater than $\sqrt{2}-1$ and therefore greater than 0.

We have also shown that $0 < a_2 < a_4 < a_6 \ldots < \sqrt{2}-1$.

Hence, $a_n > 0$ for all $n = 1, 2, 3, \ldots$

Since both sequences are monotonic and bounded, both converge.

So for the sequence $a_1, a_3, a_5, \ldots, a_{2n+1}, a_{2n+3}, \ldots$

$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} a_{2n+3} = b$.

Hence, $\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} \dfrac{1}{2+\dfrac{1}{2+a_{2n+1}}} = b$.

$$\Rightarrow \frac{1}{2+\frac{1}{2+b}} = b$$

$$\frac{b+2}{2b+5} = b$$

$$0 = 2b^2+4b-2$$

$$b = (\pm\sqrt{2})-1$$

However, the solution $b = -\sqrt{2}-1$ is less than 0, but the sequence $a_1, a_2, a_3, \ldots$ is bounded by $2/5$ from below and $1/2$ from above.

Hence, the limit is $\sqrt{2}-1$ for each sequence.

Since both sequences converge to $\sqrt{2}-1$ and the entire sequence consists solely of both subsequences, the entire sequence $a_1, a_2, a_3, \ldots$ converges to $\sqrt{2}-1$. ∎

15. Let the set $\{P_1, P_2, \ldots\}$ be the set of all interior points in S. By definition of an open set, this is a subset of S; an open ball is an open set.

Now, Suppose $\exists$ a set $A \subset S$ such that a is an open subset of E, but $A \not\subset \bigcup_{i=1}^{\infty} P_i$ given $A \neq \emptyset$

Since $A \not\subset \bigcup_{i=1}^{\infty} P_i$, A must contain a point q which is not an interior point. $\Rightarrow \nexists\ B_\varepsilon(q)\ \forall \varepsilon > 0$.

Since A is open, this is a contradiction.

18. First note that because the sequence is bounded, $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ both exist.

(for example there are infinitely many points less than the upper bound $+\varepsilon$ for $\varepsilon > 0$ and infinitely many points greater than the lower bound $-\varepsilon$ for $\varepsilon > 0$.)

Now, for the sake of contradiction, assume $\liminf_{n\to\infty} a_n > \limsup_{n\to\infty} a_n$.

However, chose any real number between them, say $p = \frac{1}{2}\left(\liminf_{n\to\infty} a_n + \limsup_{n\to\infty} a_n\right)$. Then by the pigeon hole principle, at least one of the following statements is true:

(1) There are infinitely many points in the sequence $\geq p$

(2) There are infinitely many points in the sequence $\leq p$.

However, in the case of (1), by definition of $\liminf_{n\to\infty} a_n$,

$p \geq \liminf_{n\to\infty} a_n$. Similarly in the case of (2), we have

by definition $p \leq \limsup_{n\to\infty} a_n$. Since either case yields a contradiction, we have $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$

Now we prove that equality holds if and only if a_n converges. First, note that if a_n converges to some $k \in \mathbb{R}$, then for every $\varepsilon > 0$, there exist infinitely many points inside the interval $(k-\varepsilon, k+\varepsilon)$, and only finitely many outside of the interval. Hence $\liminf_{n\to\infty} a_n \leq k+\varepsilon$ and $\limsup_{n\to\infty} a_n \geq k-\varepsilon$, for all $\varepsilon > 0$. But this exactly means that $\limsup_{n\to\infty} a_n \geq \liminf_{n\to\infty} a_n$.

Now suppose $\lim_{n\to\infty} \inf a_n = \lim_{n\to\infty} \sup a_n = X$

But this means that for any $\varepsilon > 0$, there are only finitely many points ~~[illegible]~~ greater than $\lim_{n\to\infty} \sup a_n + \varepsilon = X + \varepsilon$ and only finitely many points less than $\lim_{n\to\infty} \inf a_n - \varepsilon = X - \varepsilon$

~~[illegible]~~ But this means exactly that the sequence converges (to X). $\square$

20) $|x+iy| = \sqrt{x^2+y^2}$

(a) $d((x,y),(0,0)) = \sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2}$

(b) $|z_1+z_2| \le |z_1| + |z_2|$

$d((x_1+x_2, y_1+y_2),(0,0)) \le d((x_1,y_1),(0,0)) + d((x_2,y_2),(0,0))$

$\sqrt{(x_1+x_2)^2+(y_1+y_2)^2} \le \sqrt{(x_1^2+y_1^2)} + \sqrt{x_2^2+y_2^2}$

$\sqrt{x_1^2+2x_1x_2+x_2^2+y_1^2+2y_1y_2+y_2^2} \le \sqrt{x_1^2+y_1^2} + \sqrt{x_2^2+y_2^2}$

Squaring both sides, we get:

$\cancel{x_1^2} + 2x_1x_2 + \cancel{x_2^2} + \cancel{y_1^2} + 2y_1y_2 + \cancel{y_2^2} \le \cancel{x_1^2} + \cancel{y_1^2} + \cancel{x_2^2} + \cancel{y_2^2} + 2\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$

$\cancel{2}(x_1x_2+y_1y_2) \le \cancel{2}\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$

$x_1x_2+y_1y_2 \le \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$

This final inequality is the Schwartz inequality, which has already been proven to be true.

(c) $|z_1z_2| = |z_1| \cdot |z_2|$

$|(x_1+iy_1)\cdot(x_2+iy_2)| = (\sqrt{x_1^2+y_1^2})(\sqrt{x_2^2+y_2^2})$

$|x_1x_2 + i(x_1y_2+x_2y_1) - y_1y_2| = (\sqrt{x_1^2+y_1^2})(\sqrt{x_2^2+y_2^2})$

$\sqrt{(x_1x_2-y_1y_2)^2 + (x_1y_2+x_2y_1)^2} = (\sqrt{x_1^2+y_1^2})(\sqrt{x_2^2+y_2^2})$

$\sqrt{(x_1x_2)^2 - \cancel{2x_1y_1x_2y_2} + (y_1y_2)^2 + (x_1y_2)^2 + \cancel{2x_1y_1x_2y_2} + (x_2y_1)^2} = \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$

$(x_1x_2)^2 + (y_1y_2)^2 + (x_1y_2)^2 + (x_2y_1)^2 = (x_1x_2)^2 + (y_1x_2)^2 + (x_1y_2)^2 + (y_1y_2)^2$

22. If $d(x,y)=\|x-y\|$, then, from (i) we have $\|x-y\|\geq 0 \quad \forall x,y\in V$
$\rightarrow d(x,y)\geq 0 \quad \forall x,y \in V$

from (ii), we have $\|x-y\|=0$ if and only if $x-y=0$
$\rightarrow x=y$

$\rightarrow d(x,y)=0$ if and only if $x=y$

from (iii), we have $\|x-y\| = \|(-1)(y-x)\| = |-1|\ \|y-x\| = \|y-x\|$
$\rightarrow d(x,y)=d(y,x)$

from (iiii) $\|x-y+y-z\| \leq \|x-y\|+\|y-z\|$
$\rightarrow d(x,z)\leq d(x,y)+d(y,x)$

$\rightarrow$ V combined with $d(x,y)=\|x-y\|$ is a metric space. $\square$

Next we need to show $\sqrt{x_1^2+\dots+x_n^2}$ is a norm on $\mathbb{R}^n$.
First, since $x_1^2, x_2^2, \dots, x_n^2 \geq 0$, $\|x\|\geq 0$
Second, if $x_i^2>0$ for $i=\{1,\dots,n\}$ then $\|x\|>0$, but if
$x_i=0$ for $i=\{1,\dots,n\}$, $\|x\|=0$. Hence
$\|x\|=0$ iff $x=0$
Third, $\|cx\| = \sqrt{(cx_1)^2+(cx_2)^2+\dots+(cx_n)^2} = |c|\sqrt{x_1^2+x_2^2+\dots+x_n^2} = |c|\ \|x\|$
for $c\in\mathbb{R}$

Fourth, $\|x+y\| = \sqrt{(x_1+y_1)^2+\dots+(x_n+y_n)^2}$
$= \sqrt{x_1^2+x_2^2+\dots+x_n^2+2(x_1y_1+x_2y_2+\dots+x_ny_n)+y_1^2+y_2^2+\dots+y_n^2}$
by Schwarz inequality $\leq \sqrt{x_1^2+\dots+x_n^2+2\sqrt{x_1^2+\dots+x_n^2}\sqrt{y_1^2+\dots+y_n^2}+y_1^2+\dots+y_n^2}$
$= \sqrt{x_1^2+\dots+x_n^2}+\sqrt{y_1^2+\dots+y_n^2} = \|x\|+\|y\|$ $\square$

Note that if this norm is associated with a distance as in the first part of the problem, we are dealing with E^n

If $n=2$, then we have $\|x\|=\sqrt{x_1^2+x_2^2}$, and if this is associated with the distance $:= \|x-y\|$ then we have E^2 or if identified as in problem 20, $\mathbb{C}$.

We will now prove that $|z|$ is a norm on $\mathbb{C}$.

Let $z = x + yi \rightarrow |z| = \sqrt{x^2+y^2}$ (from prob. 20)

since this is the same as the previous norm with $n=2$, we are done. $\square$

Now we wish to show that $\|x\| = |x_1| + \dots + |x_n|$ for $x \in \mathbb{R}^n$ is a norm on $\mathbb{R}^n$

First $\|x\| \geq 0 \ \forall x \in \mathbb{R}^n$ since $|x_i| \geq 0 \ \forall x_i \in \mathbb{R}$

Second if $x=0$, then $\|x\| = |0| + |0| + \dots + |0| = 0$ and if $\|x\| = 0$,

$|x_1| = 0, |x_2| = 0, \dots, |x_n| = 0 \rightarrow x = 0$

third, $\|cx\| = |cx_1| + |cx_2| + \dots + |cx_n| = |c|(|x_1| + |x_2| + \dots + |x_n|) = |c|\,\|x\|$

Fourth, $\|x+y\| = |x_1+y_1| + \dots + |x_n+y_n|$, but $|x_i+y_i| \leq |x_i| + |y_i|$,

$\rightarrow |x_1+y_1| + \dots + |x_n+y_n| \leq |x_1| + |y_1| + \dots + |x_n| + |y_n| = \|x\| + \|y\|$ $\square$

Finally, we wish to show that $\|x\| = \max\{|x_1|, \dots, |x_n|\}$ for $x \in \mathbb{R}^n$ is a norm on $\mathbb{R}^n$

First $\|x \geq 0\|$ since $|x_i| \geq 0 \ \forall x_i \in \mathbb{R}$, $\max\{|x_1|, \dots, |x_n|\}$ is certainly ≥ 0

Second if $x=0$, $\max\{|x_1|, |x_2|, \dots, |x_n|\} = \max\{|0|, \dots, |0|\} = 0$

and if $\|x\| = 0$, $|x_1| \leq 0$, $|x_2| \leq 0, \dots, |x_n| \leq 0 \rightarrow x_1 = x_2 = \dots = x_n = 0$

$\rightarrow x = 0$

third, $\|cx\| = \max\{|cx_1|, \dots, |cx_n|\} = \max\{|c|\,|x_1|, \dots, |c|\,|x_n|\}$

$= |c| \max\{|x_1|, \dots, |x_n|\} = |c|\,\|x\|$

Fourth $\|x+y\| = \max\{|x_1+y_1|, \dots, |x_n+y_n|\} \leq \max\{|x_1| + |y_1|, \dots, |x_n| + |y_n|\}$

but since $|x_i| \leq \max\{|x_1|, \dots, |x_n|\}$ and $|y_i| \leq \max\{|y_1|, \dots, |y_n|\}$

$\rightarrow \max\{|x_1| + |y_1|, \dots, |x_n| + |y_n|\} \leq \max\{|x_1|, |x_2|, \dots, |x_n|\} + \max\{|y_1|, \dots, |y_n|\}$

$= \|x\| + \|y\|$ $\square$