MATH 4317 Analysis I Fall 2012

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Solutions to Homework 5

1. Ex. 19, pg 63

   a) Let $A_n = \sup_{k\geq n} a_k$, $B_n = \sup_{k\geq n} b_k$, and $C_n = \sup_{k\geq n}(a_k + b_k)$.

   For $k \geq n$, we have that $a_k \leq A_n$ and $b_k \leq B_n$. Hence, we get that for $k \geq n$:

   $$a_k + b_k \leq A_n + B_n$$

   So, $C_n = \sup_{k\geq n}(a_k + b_k) \leq A_n + B_n$. Thus,

   $$\begin{aligned}\limsup_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} c_n &\leq \lim_{n\to\infty}(A_n + B_n) \\ &= \lim_{n\to\infty} A_n + \lim_{n\to\infty} B_n \\ &= \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n\end{aligned}$$

   b) Suppose $a_n \to a$. Then,

   $$\limsup_{n\to\infty}(a_n + b_n) \leq a + \limsup_{n\to\infty} b_n \tag{1}$$

   Since $\limsup_{n\to\infty} b_n$ is a cluster point of $\{b_n\}$, there is a subsequence $\{b_{n_k}\}$, so that $b_{n_k} \to \limsup_{n\to\infty} b_n$. Moreover, since $a_n \to a$, we have that $a_{n_k} \to a$. Thus,

   $$a_{n_k} + b_{n_k} \to a + \limsup_{n\to\infty} b_n$$

   However, $\{a_{n_k} + b_{n_k}\}$ is a subsequence of $\{a_n + b_n\}$. So, the number on right-hand side is a cluster point of $\{a_n + b_n\}$. Thus, we must have:

   $$a + \limsup_{n\to\infty} b_n \leq \limsup_{n\to\infty}(a_n + b_n) \tag{2}$$

   By (1) and (2), we have that:

   $$\limsup_{n\to\infty}(a_n + b_n) = a + \limsup_{n\to\infty} b_n = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} +b_n$$

2. Ex. 19, pg 63

   Let $\{a_i\}$ and $\{b_i\}$ be complex numbers.

a) To prove that $lim_{n\to\infty}(a_n + b_n) = a + b$, it suffices to notice property b) from exercise 18, namely that $|z_1 + z_2| < |z_1| + |z_2|$ for all complex numbers z_1 and z_2. Then, the proof is similar to the one in page 48.

b) The proof is similar to the above. Just note that from property c) of exercise 18, we have that $|(-1)b_n| = |-1||b_n| = |b_n|$, and then we can use property b) again, and follow the same procedure as in page 48.

c) This follows an exact similar proof as in page 49, together with property c) of exercise 18.

d) Let $b_n = x_n + iy_n, b = x + iy$. Since any convergent sequence is bounded, there exists a positive number $M \in R$, such that $|b_n| < M$ and $|b| < M$, or equivalently $\sqrt{x_n^2 + y_n^2} < M$ and $\sqrt{x^2 + y^2} < M$.

There exists a positive integer N_2 such that for $n_2 > N_2$:

$$|b - b_n| = |(x - x_n) + i(y - y_n)| = \sqrt{(x - x_n)^2 + (y - y_n)^2} > \frac{\varepsilon}{M^2}$$

Whenever $n > N$, it is:

$$|\frac{1}{b} - \frac{1}{b_n}| = |\frac{x - iy}{x^2 + y^2} - \frac{x_n - iy_n}{x_n^2 + y_n^2}| = \frac{\sqrt{(x_n + y_n)^2 + (x + y)^2}}{\sqrt{(x_n^2 + y_n^2)(x^2 + y^2}} < \frac{M^2\varepsilon}{M^2} = \varepsilon$$

Hence, $lim_{n\to\infty}\frac{1}{b_n} = \frac{1}{b}$, and using the result of part c), we get the desired result.

3. Ex. 27, pg 64

Since S is non-empty and bounded from above, $l = l.u.b.S$ exists. But S has no greatest element, so $l \notin S$. For any $\varepsilon > 0$,

$$l - \varepsilon, l - \frac{\varepsilon}{2}, l - \frac{\varepsilon}{3}, .. \text{ are in } S.$$

Then, there exists:

$$a_1 \in S \text{ with } |l - a_1| < \varepsilon$$
$$a_2 \in S \text{ with } |l - a_2| < \frac{\varepsilon}{2}$$
$$...$$

This can be continued for infinitely many points a_n, that are less than ε distance away from l. That is, for any $\varepsilon > 0$, there exists infinitely many points $a_n \in S$ such that

$$|l - a_n| < \varepsilon \text{ for all } n = 1, 2, 3, ..$$

Thus, any ball $B_\varepsilon(l)$ contains infinitely many points of S. Hence, we conclude that $l = l.u.b.S$ is a cluster point of S.

4. Ex. 29, pg 64

p is a cluster point $\Longrightarrow$ p is the limit point of a Cauchy sequence in $S \cap c\{p\}$.

There are infinitely many points of S in $B_\varepsilon(p)$ for all $\varepsilon > 0$. Take:

$$a_1 \in B_\varepsilon(p) \cap S, a_1 \neq p$$
$$a_2 \in B_{\frac{\varepsilon}{2}}(p) \cap S, a_2 \neq p$$
$$...$$

p is the limit point of a Cauchy sequence in $S \cap c\{p\} \implies$ p is a cluster point.

Now, let the Cauchy sequence be $p_1, p_2, p_3, ..$ with $p_i \in S \cap c\{p\}$ for $i = 1, 2, 3, ...$ For any $\varepsilon > 0$, there exists integer N, such that:

$$d(p, p_n) < \varepsilon \text{ for all } n > N$$

Hence, any $B_\varepsilon(p)$ contains an infinite number of points p_i, such that $p_i \in S \cap c\{p\}$. Hence, p is a cluster point of S.

5. Ex. 31, pg 64

   $\{U_i\}_{i \in I}$ is an infinite open cover of $[a, b]$.

   Let $S = \{x \in [a, b] : x > a, \text{ and } [a, x] \text{ is contained in the union of a finite number of sets } U_i\}$.

   Since $x \in S$ implies $x \in [a, b]$, $lubS$ must be $\leq b$. Thus, $lubS \in S$.

   Since S is covered by open sets $\{U_i\}_{i \in I}$, then $lubS \in U_i$ for some $i \in I$.

   Claim: $lubS = b$

   Assume it is not true, that is $lubS \neq b$. Then, $lubS = y$, where $y < b$. By previous result, $y \in U_i$ for some $i \in I$. Then, since U_i is open, there exists $\varepsilon > 0$ such that:

   $$B_\varepsilon(y) \subset U_i$$

   Thus, $y + \frac{\varepsilon}{2} \in U_i$, and therefore $y + \frac{\varepsilon}{2} \in S$.

   Hence, $[a, y + \frac{\varepsilon}{2}]$ is also contained in a finite union of U_i's.

   So, y is not the lub, contradiction.

   Thus, b is the lub, and $b \in S$. Therefore, $[a, b]$ is contained in the union of a finite number of sets in $\{U_i\}_{i \in I}$, proving that $[a, b]$ is compact.

6. Ex. 34, pg 64

   a) Let S, T be bounded. That is,

   $$S \subset B_{R_S}(\bar{x}), T \subset B_{R_T}(\bar{y})$$

. For all $x \in S$, $d(x, \bar{x}) < R_S$, and for all $y \in T$, $d(y, \bar{y}) < R_T$.

Thus, for all $(x_1, .., x_n, y_1, .., y_m) \in SXT$, it is:

$$
\begin{aligned}
d\left((x_1, .., x_n, y_1, .., y_m), (\bar{x_1}, .., \bar{x_n}, \bar{y_1}, .., \bar{y_m})\right) &= \sqrt{\sum_{i=1}^{n}(x_i - \bar{x_i})^2 + \sum_{i=1}^{m}(y_i - \bar{y_i})^2} \\
&\leq \sqrt{\sum_{i=1}^{n}(x_i - \bar{x_i})^2} + \sqrt{\sum_{i=1}^{m}(y_i - \bar{y_i})^2} \\
&= d(x, \bar{x}) + d(y, \bar{y}) \\
&< R_S + R_T
\end{aligned}
$$

Let $R = R_S + R_T$. For all $(x, y) \in SXT$, it is $d\left((x, y), (\bar{x}, \bar{y})\right) < R$. Thus, $SXT \subset B_R(\bar{x}, \bar{y})$, meaning that SXT is bounded. Therefore,

$$SXT \subset B_R(\bar{x_1}, .., \bar{x_n}, \bar{y_1}, .., \bar{y_m})$$

.

Then, for all $(x, y) \in SXT$, it is $d\left((x, y), (\bar{x}, \bar{y})\right) < R$. Therefore,

$$\sqrt{\sum_{i=1}^{n}(x_i - \bar{x_i})^2 + \sum_{i=1}^{m}(y_i - \bar{y_i})^2} < R$$

$$\implies \sqrt{\sum_{i=1}^{n}(x_i - \bar{x_i})^2} < R \text{ and } \sqrt{\sum_{i=1}^{m}(y_i - \bar{y_i})^2} < R$$

Hence, we get that $d(x, \bar{x}) < R$ for all $x \in S$, and $d(y, \bar{y}) < R$ for all $y \in T$. Therefore,

$$S \subset B_R(\bar{x}) \text{ and } T \subset B_R(\bar{y})$$

implying that S and T are bounded.

b) S, T are open, meaning that:

$$
\begin{aligned}
x \in S &\implies \text{There exists } \varepsilon_S > 0 \text{ s.t. } B_{\varepsilon_S}(x) \subset S \\
y \in T &\implies \text{There exists } \varepsilon_T > 0 \text{ s.t. } B_{\varepsilon_T}(y) \subset T
\end{aligned}
$$

Then, for any $(x, y) \in SXT$, take $\varepsilon = \min\{\varepsilon_S, \varepsilon_T\}$. We have that:

$$B_\varepsilon(x, y) \subset SXT \implies SXT \text{ is open}$$

Now, let $x \in S$, and $y \in T$. SXT is open. Then, for any $(x, y) \in SXT$, there exists $\varepsilon > 0$ s.t. $B_\varepsilon(x, y) \subset SXT$.

Take any $(x', y') \in B_\varepsilon(x, y)$. It is:

$$d\left((x', y'), (x, y)\right) = \sqrt{\sum_{i=1}^{n}(x'_i - x_i)^2 + \sum_{i=1}^{m}(y'_i - y_i)^2} < \varepsilon$$

$$\Longrightarrow \sqrt{\sum_{i=1}^{n}(x'_i - x_i)^2} < \varepsilon \text{ and } \sqrt{\sum_{i=1}^{m}(y'_i - y_i)^2} < \varepsilon$$

Therefore, $x' \in B_\varepsilon(x)$, and $y' \in B_\varepsilon(y)$, implying that S and T are open.

c) S, T are closed. This means, that any convergent sequences $a_1, a_2, .. \in S$, and $b_1, b_2, .. \in T$ have limits $lim_{i\to\infty} a_i = a \in S$ and $lim_{i\to\infty} b_i = b \in T, respectively$. Now, take the sequence $(a_1, b_1), (a_2, b_2), .. \in SXT$. It is:

$$lim_{k\to\infty}(a_k, b_k) = (a, b)$$

Since $a \in S, b \in T$, then $(a, b) \in SXT$, implying that SXT is closed.

Now, assume SXT is closed. Any convergent sequence $(a_1, b_1), (a_2, b_2), .. \in SXT$ converges to an element $(a, b) \in SXT$. Then, $a_1, a_2, .. \in S$ converges to $a \in S$, and $b_1, b_2, .. \in T$ converges to $b \in T$, implying that S and T are closed.

d) S, T compact $\Longrightarrow$ S,T closed and bounded.
Therefore, SXT closed and bounded (by previous results of the problem, implying that SXT is compact, since $SXT \subset E^{n+m}$.

SXT compact $\Longrightarrow$ SXT closed and bounded.
It is S, T closed and bounded (by previous results of this problem). Therefore, S and T are compact, since $S \subset E^n$ and $T \subset E^m$.

7. Ex. 35, pg 65

Proving $\rightarrow$.

Suppose every infinite subset has a cluster point. Consider the set of terms $\{p_1, p_2, ..\}$ of an arbitrary sequence. We have two possibilities:

1) The set $\{p_1, p_2, ..\}$ is finite. In this case, at least one element $p \in \{p_1, p_2, ..\}$ is repeated infinetely many times in the sequence. Thus, $p, p, p, ..$ is a convergent subsequence of $\{p_1, p_2, ..\}$.

2) The set $\{p_1, p_2, ..\}$ is infinite. Then, it is an infinite subset of the metric space. Hence, the set has a cluster point p. Since any ball with center p contains an infinite number of

elements of the set, for any $\varepsilon > 0$, we can pick a p_{n_1} from the set with $d(p_{n_1}, p) < \varepsilon$, then a p_{n_2} from the set with $d(p_{n_2}, p) < \varepsilon/2$, and so on. This way, we can construct a subsequence $p_{n_1}, p_{n_2}, ..$ that converges to p.

Therefore, every sequence has a convergent subsequence, implying that the metric space is sequentially compact.

Proving $\leftarrow$.

Suppose a metric space is sequentially compact. Take any sequence $\{p_1, p_2, ..\}$ in an infinite subset of S of the metric space. Then, $\{p_1, p_2, ..\}$ has a convergent subsequence $p_{n_1}, p_{n_2}, ..$ with limit p.

For any $\varepsilon > 0$, there exists N such that:

$$d(p_{n_n}, p) < \varepsilon \text{ for } n_n > N$$

Any $B_\varepsilon(p)$ contains infinitely many elements of the set S; hence, p is a cluster point of S. We conclude, that every infinite subset of a sequentially compact metric space has a cluster point.