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Math 4317
Homework - Week 7

a) Discuss the continuity of the function $f: \mathbb{R} \to \mathbb{R}$ if

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

f is continuous at p_0 if for any $\varepsilon > 0$, $\exists\ \delta > 0$ so that if $p \in E$ and $d(p, p_0) < \delta$, then $d'(f(p), f(p_0)) < \varepsilon$.

Assume $p \in E$ and $p < 0$. Then $d(p, p_0) = |p - p_0|$ and

$$d'(f(p), f(p_0)) = d'(0 - 0) = 0.$$

So given any $\varepsilon > 0$, δ can be any positive number.

Now assume $p \in E$ and $p \geq 0$. Because $f(p) = p$ for all $p \geq 0$,

$$d(f(p), f(p_0)) = d(p, p_0),$$

and for each $p_0 \in E$, given $\varepsilon > 0$, we can choose $\delta = \varepsilon$.

So $f(x)$ is continuous for all x.

d) Discuss the continuity of the function $f: \mathbb{R} \to \mathbb{R}$ if

$$f(x) = \begin{cases} 0 & x \text{ is not rational} \\ \frac{1}{q} & x = \frac{p}{q},\ p, q \text{ are integers w/ no common divisors other than } \pm 1, \text{ and } q > 0. \end{cases}$$

Let $x_0 \in \mathbb{R}$.

There must exist a rational number $\frac{p_0}{q_0} \in B_\delta(x_0)$ for any $\delta > 0$.

There exists a positive integer N such that

$$\frac{p_0}{q_0} + \frac{x_0}{N} \in B_\delta(x_0).$$

Suppose x_0 is irrational. Then

$$|f(x) - f(x_0)| = |f(x) - 0| = |f(x)| < \varepsilon.$$

Because $x = \frac{p}{q}$, $\quad |f(x)| = \frac{1}{q} < \varepsilon.$

So if we have a $\varepsilon \leq \frac{1}{q}$, there is a contradiction.

Suppose $x_0 = \frac{p_0}{q_0}$, and

$$|f(x) - f(x_0)| = \left| f(x) - \frac{1}{q_0} \right| < \varepsilon.$$

Letting x be irrational,

$$\left| f(x) - \frac{1}{q} \right| = \frac{1}{q} < \varepsilon,$$

so if $\varepsilon \geq \frac{1}{q}$ we have the same contradiction (because ε must be any positive real number).

Thus f is not continuous for all $x \in \mathbb{R}$.

2) Let E, E' be metric spaces, $f: E \to E'$ a continuous function. Show that if S is a closed subset of E' then $f^{-1}(S)$ is a closed subset of E. Derive from this the results that if f is a continuous real-valued function on E then the sets $\{p \in E : f(p) \leq 0\}$, $\{p \in E : f(p) \geq 0\}$, $\{p \in E : f(p) = 0\}$ are closed.

Claim: $f^{-1}(S)$ is a closed subset of E.

Proof.

It is known that $S \subset E'$ is closed, so $\complement S \subset E'$ is open.

Because f is continuous, it is true that for every open subset U of E', the inverse image $f^{-1}(U) = \{p \in E : f(p) \in U\}$ is an open subset of E.

So $f^{-1}(\complement S) \subset E$ is open.

It follows that $\complement(f^{-1}(S)) = \{p \in E : f(p) \in \complement S\} = f^{-1}(\complement S)$, which is open.

so $f^{-1}(S)$ is closed.

The sets $\{x \in \mathbb{R} : x \leq 0\}$, $\{x \in \mathbb{R} : x \geq 0\}$, $\{0\}$ are each closed subsets of $\mathbb{R}$.

So when $f: E \to \mathbb{R}$,

$\{p \in E : f(p) \leq 0\}$, $\{p \in E : f(p) \geq 0\}$, and $\{p \in E : f(p) = 0\}$

are the inverse images of these closed subsets.

As proven above, these subsets must then be closed subsets of metric space E.

(4) U, V are open/closed intervals in $\mathbb{R}$

$f: U \to V$ is strictly increasing and onto

Want to show that f is continuous.

U, V are intervals so they are bounded and if U, V are closed then U, V are compact subsets of $\mathbb{R}$

Assume that f is not continuous, so it does not attain a maximum or minimum at some point $u \in U$

Since V is nonempty compact let $a = \text{lub } V$, $b = \text{glb } V$ $a, b \in V$ and f is onto so $\exists$ $a', b' \in U$ sth $f(a') = a$ and $f(b') = b$. Since ~~is~~ f is onto this implies that

$f(b') \leq f(u)$ $f(a') \geq f(u)$ for all $u \in U$

$\otimes$ since f does not attain a maximum or minimum

$\therefore$ f is continuous.

If U, V are open, assume f is not continuous. So $\exists$ some open subset $B \subseteq V$ sth $f^{-1}(B) = \{u \in U : f(u) \in B\}$ is not an open subset of U. Let $\bar{u} \in f^{-1}(B)$ Then $f(\bar{u}) \in B$ but since B is open $\exists$ $\varepsilon > 0$ sth $|f(u) - f(\bar{u})| < \varepsilon$ for all $u \in U$. But if we choose a $\delta > 0$ sth $|u - \bar{u}| < \delta$ then $|f(u) - f(\bar{u})| < \varepsilon$ where $u \in f^{-1}(B)$ implying that $f^{-1}(B)$ is open $\otimes$ since earlier $f^{-1}(B)$ was stated as not open so f is continuous.

U, V are bounded and onto so they are $1-1$ and a similar proof can be used to show that f^{-1} is continuous

(9) a) Prove $\sqrt{x}$ is continuous on $\{x \in \mathbb{R} : x \geq 0\} = E$

Pick $x' \in E$. If f is continuous on E then for every $\varepsilon > 0$ we need to find $\delta > 0$ s.t.h. if $x \in E$ and $|x - x'| < \delta$ then

$$|f(x) - f(x')| = |\sqrt{x} - \sqrt{x'}| < \varepsilon$$

$$|\sqrt{x} - \sqrt{x'}| = \left| (\sqrt{x} - \sqrt{x'}) \frac{(\sqrt{x} + \sqrt{x'})}{(\sqrt{x} + \sqrt{x'})} \right| = \left| \frac{x - x'}{\sqrt{x} + \sqrt{x'}} \right| \leq \frac{|x - x'|}{\sqrt{x'}}$$

Let $\frac{|x - x'|}{\sqrt{x'}} < \varepsilon$ $\quad |\sqrt{x} - \sqrt{x'}| < \varepsilon \quad |x - x'| < \varepsilon \sqrt{x'} = \delta$

so $\sqrt{x}$ is continuous on E

b) $\lim_{x \to 1} \frac{x - 1}{\sqrt{x} - 1}$ $\{1\}$ is a cluster point. Let $f : C\{1\} \to \mathbb{R}$

and $f(x) = \frac{x - 1}{\sqrt{x} - 1}$ $\quad \frac{x - 1}{\sqrt{x} - 1} \frac{(\sqrt{x} + 1)}{(\sqrt{x} + 1)} = \frac{(x - 1)(\sqrt{x} + 1)}{(x - 1)} = \sqrt{x} + 1$ and so

$\lim_{x \to 1} \frac{x - 1}{\sqrt{x} - 1} = \lim_{x \to 1} \sqrt{x} + 1 = \lim_{x \to 1} \sqrt{1} + 1 = 2$ can be evaluated

since $\sqrt{x}, 1$ are continuous.

c) $\lim_{x \to +\infty} \frac{x}{2x^2 + 1}$ can be defined as $\lim_{y \to 0} g(y)$ $\quad g : (0,1) \to \mathbb{R}$

$$g(y) = f(1/y) = \frac{1/y}{2(1/y^2) + 1} = \frac{y}{2 + y^2} \quad \lim_{y \to 0} g(y) = \frac{y}{2 + y^2} = 0$$

So this means that for every $\varepsilon > 0$, there is $\delta > 0$ s.t.h. if $y \in (0,1)$ and $|y| < \delta$ then $\left| y / 2 + y^2 \right| < \varepsilon$

$$\left| \frac{y}{2 + y^2} \right| = \frac{|y|}{|2 + y^2|} \Rightarrow \frac{|y|}{2 + y^2} \leq |y| < \delta \quad \therefore \text{ if we let } \varepsilon = \delta$$

$|y| < \varepsilon$ then $\left| \frac{y}{2 + y^2} \right| < \varepsilon$

(10) a) $f(x,y) = \begin{cases} \frac{1}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Using the prop on pg 74:

Consider the sequence $(\frac{1}{n}, 0)$: $\lim_{n\to 0} (\frac{1}{n}, 0) = (0,0)$

If f is continuous, at (0,0) $f(0,0) = 0$ but

$\lim_{n\to\infty} f(\frac{1}{n}, 0) = n^2$ so the limit does not exist and f is not continuous at (0,0)

b) $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Again: $\lim_{n\to\infty} (\frac{1}{n}, \frac{1}{n}) = (0,0)$ but $\lim_{n\to\infty} f(\frac{1}{n}, \frac{1}{n}) = \frac{\frac{1}{n^2}}{\frac{2}{n^2}} = \frac{1}{2}$

which is not $f(0,0) = 0$ so f is not continuous at (0,0)

c) $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Again need to check continuity at (0,0)

f(x,y) is continuous if $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ sth if

$d((x,y),(0,0)) < \delta$ then $\left|\frac{xy^2}{x^2+y^2}\right| < \varepsilon$

$\left((x-y)^2 = x^2 - 2xy + y^2 \Rightarrow x^2+y^2 \geq 2xy\right)$

$\left|\frac{xy^2}{x^2+y^2}\right| = \frac{|xy^2|}{|x^2+y^2|} \leq \frac{|xy^2|}{|2xy|} \Rightarrow \left|\frac{y^2}{2y}\right| = \left|\frac{y}{2}\right| < \varepsilon$ so $|y| < 2\varepsilon$

and $|x| < 2\varepsilon$ by assumption $\Rightarrow x^2+y^2 < 8\varepsilon^2$

$\Rightarrow d((x,y),(0,0)) = \sqrt{x^2+y^2} < \sqrt{8\varepsilon^2} < 2\sqrt{2}\,\varepsilon$

$\therefore$ f is continuous at (0,0)

11) **Proposition.** Let f and g be real-valued functions on a metric space E. If f and g are continuous at a point $p_0 \in E$, then so are the functions $f+g$, $f-g$, fg and $\frac{f}{g}$, the last under the proviso that $g(p_0) \neq 0$.

Claim: $f+g$ is continuous at $p_0 \in E$.

Proof.

Due to the continuity of f, g at p_0, $\exists\ \delta_1, \delta_2$ such that if $p \in E$ and

$$d(p, p_0) < \min\{\delta_1, \delta_2\},$$

then
$$|f(p) - f(p_0)| < \frac{\varepsilon}{2} \quad \text{and} \quad |g(p) - g(p_0)| < \frac{\varepsilon}{2}.$$

so
$$|(f+g)(p) - (f+g)(p_0)| = |f(p) - f(p_0) + g(p) - g(p_0)|$$
$$\leq |f(p) - f(p_0)| + |g(p) - g(p_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

for all $p \in E$. So $f+g$ is continuous at p_0.

Claim: $f-g$ is continuous at $p_0 \in E$.

proof:

once again, $\exists\ \delta_1, \delta_2$ such that if $p \in E$ and

$$d(p, p_0) < \min\{\delta_1, \delta_2\}, \text{ then}$$
$$|f(p) - f(p_0)| < \frac{\varepsilon}{2} \quad \text{and} \quad |g(p) - g(p_0)| < \frac{\varepsilon}{2}.$$

so
$$|(f-g)(p) - (f-g)(p_0)| = |(f(p) - f(p_0)) + -(g(p) - g(p_0))|.$$
$$\leq |f(p) - f(p_0)| + |-(g(p) - g(p_0))|$$
$$= |f(p) - f(p_0)| + |g(p_0) - g(p)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $f-g$ is continuous at p_0.

Claim: fg is continuous at $p_0 \in E$.

proof:

Letting $\delta = \min\{\delta_1, \delta_2\}$, where $|p - p_0| < \delta$,

$$|g(p) - g(p_0)| < \varepsilon, \quad \text{and} \quad |g(p)| < |g(p_0)| + \varepsilon.$$

Then
$$|(fg)(p) - (fg)(p_0)| = |f(p)g(p) - f(p_0)g(p_0)|$$
$$= |(f(p) - f(p_0))(g(p)) + (g(p) - g(p_0))(f(p_0))|$$
$$\leq |f(p) - f(p_0)||g(p)| + |g(p) - g(p_0)||f(p_0)|$$
$$\leq |f(p) - f(p_0)|(|g(p_0)| + \varepsilon) + |g(p) - g(p_0)||f(p_0)|.$$

Taking $|f(p)-f(p_0)| < \frac{\varepsilon}{2(|g(p_0)|+\varepsilon)}$ and $|g(p)-g(p_0)| < \min\left\{\varepsilon, \frac{\varepsilon}{2|f(p_0)|}\right\}$

$$|(fg)(p)-(fg)(p_0)| < \frac{\varepsilon}{2(|g(p_0)|+\varepsilon} (|g(p_0)|+\varepsilon) + \frac{\varepsilon}{2|f(p_0)|}|f(p_0)|$$

$$= \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

so fg is continuous at p_0.

Claim: $\frac{f}{g}$ is continuous at $p_0 \in E$, given $g(p_0) \neq 0$.

Proof. suppose $|g(p)-g(p_0)| < \min\left\{\frac{|g(p_0)|}{2}, \frac{|g(p_0)|^2\varepsilon}{2}\right\}$ where $d(p,p_0) < \delta$.

Then

$$|g(p)| = |g(p_0)-(g(p_0)-g(p))| \geq |g(p_0)| - |g(p)-g(p_0)|$$
$$> |g(p_0)| - \frac{|g(p_0)|}{2} = \frac{|g(p_0)|}{2}.$$

so

$$\left|\frac{1}{g(p)} - \frac{1}{g(p_0)}\right| = \frac{|g(p)-g(p_0)|}{|g(p)||g(p_0)|} < \frac{|g(p_0)|^2\frac{\varepsilon}{2}}{|g(p_0)|\cdot\frac{|g(p_0)|}{2}} = \varepsilon.$$

Because $\frac{1}{|g(p)|} < \frac{1}{\frac{|g(p_0)|}{2}}$, the function $\frac{1}{g}$ is continuous at p_0.

So $f\left(\frac{1}{g}\right) = \frac{f}{g}$ will be continuous at $p_0 \in E$ because fg is continuous (proven above).