CH. 8 HOMEWORK

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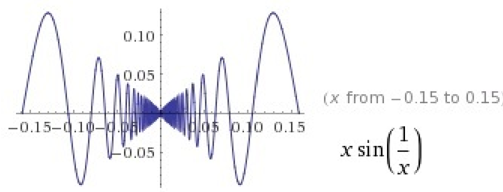
(1) Problem 1

Discuss the continuity of the function $f : \mathbb{R} \to \mathbb{R}$ if

(b) $f(x) = \begin{cases} xsin(\frac{1}{x}) & \text{for} \quad x \neq 0 \\ 0 & \text{for} \quad x = 0 \end{cases}$



Proof. We wish to show f is continuous at all x. First we will show f is continuous at $x = 0$. Let $\epsilon > 0$ be given. Consider,

$$|f(x) - f(0)| = \left| xsin\left(\frac{1}{x}\right) - 0 \right| = \left| xsin\left(\frac{1}{x}\right) \right| = |x| \left| sin\left(\frac{1}{x}\right) \right| \leq |x| < x.$$

If we let $\epsilon = \delta$, then it follows that when $|x - 0| < \delta$, we have $|f(x) - f(0)| < \epsilon$. Therefore f is continuous at $x = 0$.

Now we wish to show f is continuous when $x \neq 0$. Consider,

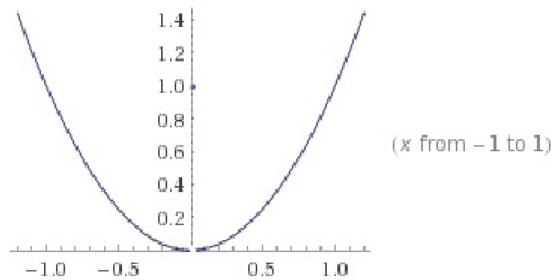
$$\begin{aligned} |f(x) - f(x_o)| &= \left| xsin\left(\frac{1}{x}\right) - x_o sin\left(\frac{1}{x_o}\right) \right| \\ &= \left| xsin\left(\frac{1}{x}\right) - xsin\left(\frac{1}{x_o}\right) + xsin\left(\frac{1}{x_o}\right) - x_o sin\left(\frac{1}{x_o}\right) \right| \\ &= \left| x\left(sin\left(\frac{1}{x}\right) - sin\left(\frac{1}{x_o}\right)\right) + sin\left(\frac{1}{x_o}\right)(x - x_o) \right| \\ &\leq \left| x\left(sin\left(\frac{1}{x}\right) - sin\left(\frac{1}{x_o}\right)\right) \right| + \left| sin\left(\frac{1}{x_o}\right)(x - x_o) \right| \\ &< |x| \left| sin\left(\frac{1}{x}\right) - sin\left(\frac{1}{x_o}\right) \right| + \left| sin\left(\frac{1}{x_o}\right) \right| |x - x_o|. \end{aligned}$$

Since $\left|sin\left(\frac{1}{x}\right)\right| \leq 1$, we know that $\left|sin\left(\frac{1}{x}\right) - sin\left(\frac{1}{x_o}\right)\right| \leq 1$. Then it follows that,

$$\begin{aligned} |f(x) - f(x_o)| &< |x| \left|sin\left(\frac{1}{x}\right) - sin\left(\frac{1}{x_o}\right)\right| + \left|sin\left(\frac{1}{x_o}\right)\right| |x - x_o| \\ &< |x| + |x - x_o| \end{aligned}$$

Let $\epsilon > |x| + \delta$, then when $|x - x_o| < \delta$, it follows that $|f(x) - f(x_o)| < \epsilon$. Therefore f is continuous at $x \neq 0$. □

(c) $f(x) = \begin{cases} x^2 & \text{for} \quad x \neq 0 \\ 1 & \text{for} \quad x = 0 \end{cases}$



Proof. Let $x_n = \frac{1}{n}$, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$$\begin{aligned} &\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (\tfrac{1}{n})^2 \\ &= \lim_{n\to\infty} \tfrac{1}{n^2} = 0 \neq f(0) = 1 \end{aligned}$$

Thus $f(x)$ is not continuous by proposition on page 74.

Another way to prove that $f(x)$ is discontinuous at $x = 0$ is again fairly straightforward. Similar to example one on page 69, we see that all we need to show is that for any $\epsilon > 0$ we can not find a $\delta > 0$, such that $|f(x) - f(0)| < \epsilon$, whenever $|x - 0| < \delta$.

$$\begin{aligned} |x^2 - 1^2| &= |(x+1)(x-1)| \\ &= |(x - 1 + 1 + 1)(x - 1)| \\ &< (|x - 1| + |2|)|x - 1| \end{aligned}$$

Notice for $|f(x) - f(0)| < \epsilon$ we would have to have $|x - 1| < \delta$, but we also need $|x - 0| < \delta$. Which is a contradiction, hence f is not continuous at $x = 0$ and is therefore discontinuous. □

(2) Problem 14

(a) Prove the if S is a nonempty compact subset of a metric space E and $p_o \in E$ then $\min\{d(p_o, p) : p \in S\}$ exists. ("distance from p_o to S")

Proof. Let S be a nonempty compact subset of E, and $p_o \in E$. Let $f : S \to \mathbb{R}$ where $f(p) = d(p_o, p)$. We wish to show f is continuous, since continuous real valued functions on nonempty compact metric space attain a minimum value. Consider $p_1 \in S$ and $\epsilon > 0$ given,

$$|f(p) - f(p_1)| = |d(p_o, p) - d(p_o, p_1)| = |d(p, p_o) - d(p_o, p_1)| \leq d(p, p_1).$$

If you let $d(p, p_1) < \epsilon$, then it follows that $|f(p) - f(p_1)| < \epsilon$. So let $\delta = \epsilon$. Then we can conclude that f is continuous. Hence we can say f attains a min at some point, which means $\min\{d(p_o, p) : p \in S\}$ exists. □

(b) Prove that if S is a nonempty closed subset of E^n and $p_o \in E^n$ then $\min\{d(p_o, p) : p \in S\}$ exists.

Proof. Let E, E' be metric spaces, and $f : E \to E'$ be a continuous, one-to-one, and onto function. We wish to show $f^{-1} : E' \to E$ is continuous by showing that for every closed set $A \in E$, $f(A) \in E'$ is closed. Assume we have a convergent subsequence $q_n \in f(A)$ We have that $f(A) \in E'$ From one to one and onto we know that $\exists p_n : f(p_n) = q_n, p_n \in A$. Since $\lim_{n\to\infty} q_n = q_o$ we know that $\exists p_o \in E$, since we have one to one and onto$f(p_o) = q_o$. Thus $f(p_n) = q_n \to q_o = f(p_o).f(p_o) = q_o$. So we know that $\lim_{n\to\infty} p_n - p_o$ since A is closed. $p_o \in A$sp$f(p_o) = q_o) \in f(A)$ therefore $f(A)$ is closed. Thus we have that closed sets map to closed sets and if we take the compliments, we see that open sets map to open sets. We can now conclude that $f^{-1} : E' \to E$ is continuous □

(3) Problem 15

Prove that for any nonempty compact metric space E, $max\{d(p, q) : p, q \in E\}$ exists ("diameter of E"). (Hint: Start with a sequence of pairs of points $(p_n, q_n)_{n=1,2,3,...}$ of E such that $\lim_{n\to+\infty} d(p_n, q_n) = l.u.b.\{d(p, q) : p, q \in E\}$ and pass to convergent subsequences)

Proof. Suppose there is a sequence of points $(p_n, q_n)_{n=1,2,3,...} \subset E$ such that

$$\lim_{n\to\infty} d(p_n, q_n) = L.U.B\{d(p, q) : p, q \in E\}.$$

Since E is compact, which implies E is bounded, we can say $(p_n, q_n)_{n=1,2,3,...}$ is also bounded. Another property of compactness allows us to state that there exists subsequences in E that also converge in E, i.e. $(p_{n_k}) \to p_o, (q_{n_k}) \to q_o$ as $k \to \infty$, where $p_o, q_o \in E$. This implies that $\lim_{k\to\infty} d(p_{n_k}, q_{n_k}) = d(p_o, q_o)$. Since limits are unique and a sequence that converges also has subsequences that converge to the same limit, we can say

$$\lim_{k\to\infty} d(p_{n_k}, q_{n_k}) = \lim_{n\to\infty} d(p_n, q_n) = L.U.B.\{d(p, q) : p, q \in E\}.$$

Since (p_n, q_n) is bounded, we know the L.U.B. is contained in the set $\{d(p, q) : p, q \in E\}$, which means the set has a maximum. Hence max $\{d(p, q) : p, q \in E\}$ exists. □

(4) Problem 16

Let E, E' be metric spaces, $f : E \to E'$ is a continuous function. Prove that if E is compact and f is one-one onto then $f^{-1} :: E \to E'$ is continuous. (Hint: f sends closed sets onto closed sets, therefore open sets onto open sets.)

Proof. Let E, E' be metric spaces, and $f : E \to E'$ be a continuous, one-to-one, and onto function. We wish to show $f^{-1} : E' \to E$ is continuous by showing that for every open set $U \in E$, $f(U) \in E'$ is open. Since f is continuous we know every open set $U \in E'$ maps to an open set $f^{-1}(U) \in E$. Also, since U^C is closed, we can say $f^{-1}(U^C) = (f^{-1}(U))^C$ is also closed. Hence open sets map to open sets, and closed sets map to closed sets. We know that since f is a one-to-one and onto function we can say every open (or closed) set $f(U) \in E'$ maps to an open (or closed) set $U \in E$. Hence f^{-1} is continuous. □

Alternative proof for Problem 16

Proof. Let E, E' be metric spaces, and $f : E \to E'$ be a continuous, one-to-one, and onto function. We wish to show $f^{-1} : E' \to E$ is continuous by showing that for every closed set $A \in E$, $f(A) \in E'$ is closed. Assume we have a convergent subsequence $q_n \in f(A) \subset E'$. Since f is one to one and onto we know that there exits a sequence $p_n \in A$ such that $f(p_n) = q_n$, where $q_n \in f(A)$. Let $\lim_{n\to\infty} q_n = q_o$, then we know that there exists $p_o \in E$ such that $f(p_o) = q_o$, since$f$ is one to one and onto. Thus $f(p_n) = q_n$ implies that $f(p_o) = q_o$ Since A is closed, we know that $\lim_{n\to\infty} p_n = p_o$, and $p_o \in A$. Hence $f(p_o) = q_o \in f(A)$, and therefore $f(A)$ is closed. Thus we have that closed sets map to closed sets and if we take the compliments, we see that open sets map to open sets. We can now conclude that $f^{-1} : E' \to E$ is continuous □