Homework #9

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17. Is the function x^2 uniformly continuous on $\mathbb{R}$? The function $\sqrt{|x|}$? Why?

Suppose x^2 is uniformly continuous on $\mathbb{R}$
for any $\varepsilon > 0$ $\exists$ $\delta > 0$ such that if $x_1, x_2 \in \mathbb{R}$
and $|x_1 - x_2| < \delta$ then $|x_1^2 - x_2^2| < \varepsilon$

let $x_2 = x_1 + \frac{\delta}{2}$

$$|x_1 - x_2| = |x_1 - x_1 - \frac{\delta}{2}| = \frac{\delta}{2} < \delta$$

therefore $|x_1^2 - x_2^2| < \varepsilon$

$$|x_1^2 - (x_1 + \frac{\delta}{2})^2| = |x_1^2 - x_1^2 - x_1\delta - \frac{\delta^2}{4}|$$

$$= |-x_1\delta - \frac{\delta^2}{4}| = |x_1\delta + \frac{\delta^2}{4}| < \varepsilon$$

this must be true for any $x_1 \in \mathbb{R}$ if x^2 is uniformly continuous

let $x_1 = \frac{\varepsilon}{\delta}$

$$|x_1\delta + \frac{\delta^2}{4}| = \frac{\varepsilon}{\delta}\delta + \frac{\delta^2}{4} = \varepsilon^2 + \frac{\delta^2}{4} \not< \varepsilon$$

a contradiction

conclude that x^2 is not uniformly continuous

Proof that $f(x) = \sqrt{|x|}$ is uniformly continuous

$x_1, x_2 \in \mathbb{R}$

$$|f(x_1) - f(x_2)| = |\sqrt{|x_1|} - \sqrt{|x_2|}| = \left|\frac{(\sqrt{|x_1|} - \sqrt{|x_2|})(\sqrt{|x_1|} + \sqrt{|x_2|})}{\sqrt{|x_1|} + \sqrt{|x_2|}}\right|$$

$$= \frac{|\sqrt{|x_1|}^2 - \sqrt{|x_2|}^2|}{|\sqrt{|x_1|} + \sqrt{|x_2|}|} = \frac{||x_1| - |x_2||}{\sqrt{|x_1|} + \sqrt{|x_2|}} \leq \frac{|x_1 - x_2|}{|\sqrt{|x_1|} - \sqrt{|x_2|}|}$$

note: Reverse triangle inequality: $||x_1| - |x_2|| \leq |x_1 - x_2|$

$$|\sqrt{|x_1|} - \sqrt{|x_2|}| \leq |\sqrt{|x_1|}| + |\sqrt{|x_2|}| = \sqrt{|x_1|} + \sqrt{|x_2|}$$

therefore $\frac{1}{\sqrt{|x_1|} + \sqrt{|x_2|}} \leq \frac{1}{|\sqrt{|x_1|} - \sqrt{|x_2|}|}$

$|\sqrt{|x_1|} - \sqrt{|x_2|}|^2 \leq |x_1 - x_2|$

$|\sqrt{|x_1|} - \sqrt{|x_2|}| \leq (|x_1 - x_2|)^{1/2} < \varepsilon$ whenever $|x_1 - x_2| < \varepsilon^2 = \delta$

therefore for any $\varepsilon > 0$ $\exists$ $\delta = \varepsilon^2 > 0$ such that if $x_1, x_2 \in \mathbb{R}$ and $|x_1 - x_2| < \delta$ then $|f(x_1) - f(x_2)| = |\sqrt{|x_1|} - \sqrt{|x_2|}| < \varepsilon$

therefore $\sqrt{|x|}$ is uniformly continuous on $\mathbb{R}$

18. Prove that for any metric space E, the identity function on E is uniformly continuous.

Let E be an arbitrary metric space and $f: E \to E$ is given by $f(p) = p$ for all $p \in E$

for any $\varepsilon > 0$ $|f(p_1) - f(p_2)| < \varepsilon$ whenever $|p_1 - p_2| < \delta = \varepsilon$ $\forall$ $p_1, p_2 \in E$

$|f(p_1) - f(p_2)| = |p_1 - p_2| < \varepsilon$ whenever $|p_1 - p_2| < \varepsilon$. and $p_1, p_2 \in E$

conclude that the identity function on E is uniformly continuous

19. Prove that for any metric space E and any $p_0 \in E$, the real-valued function sending any p into $d(p_0, p)$ is uniformly continuous.

$f: E \to \mathbb{R}$ and $f(p) = d(p_0, p)$

from the triangle inequality we know that

$d(p_0, p_1) \leq d(p_0, p_2) + d(p_2, p_1)$

$d(p_0, p_1) - d(p_0, p_2) \leq d(p_2, p_1) = d(p_1, p_2)$

similarly

$d(p_0, p_2) \leq d(p_0, p_1) + d(p_1, p_2)$

$d(p_0, p_2) - d(p_0, p_1) \leq d(p_1, p_2)$

therefore $|d(p_0, p_1) - d(p_0, p_2)| \leq d(p_1, p_2)$ $\forall$ $p_1, p_2 \in E$

$|f(p_1) - f(p_2)| \leq d(p_1, p_2) < \varepsilon$ whenever $d(p_1, p_2) < \varepsilon$

therefore given any real number $\varepsilon > 0$, there exists a real number $\delta = \varepsilon > 0$ such that if $p_1, p_2 \in E$ and $d(p_1, p_2) < \delta$ then $|f(p_1) - f(p_2)| < \varepsilon$

conclude that $d(p_0, p) = f(p)$ is uniformly continuous

20) State precisely and prove: A uniformly continuous function of a uniformly continuous function is uniformly continuous.

Suppose $f: E \to E'$ is uniformly continuous and $g: E' \to E''$ is uniformly continuous E, E', E'' metric spaces with metrics d, d', d'' respectively

proof that $g \circ f$ is uniformly continuous:

Since f is uniformly continuous

given any $\varepsilon_1 > 0$ $\exists$ $\delta' > 0$ such that if $p, q \in E$ and $d(p,q) < \delta$ then $d'(f(p), f(q)) < \varepsilon_1$

$f(p), f(q) \in E'$

given any $\varepsilon > 0$ $\exists$ $\delta' > 0$ such that if $f(p), f(q) \in E'$ and $d'(f(p), f(q)) < \delta''$ then $d''(g(f(p)), g(f(q))) < \varepsilon$

fix $\varepsilon_1 < \delta'$

then $\exists$ $\delta > 0$ such that

if $p, q \in E$ and $d(p,q) < \delta$ then $d'(f(p), f(q)) < \varepsilon_1 < \delta'$

then $d''(g \circ f(p), g \circ f(q)) < \varepsilon$

therefore given any $\varepsilon > 0$ $\exists$ $\delta > 0$ such that

if $p, q \in E$ and $d(p,q) < \delta$ then $d''(g \circ f(p), g \circ f(q)) < \varepsilon$

conclude that $g \circ f$ is uniformly continuous problem 21. Let S be a subset of the metric space E with the property that each point of $\complement S$ is a cluster point of S (one calls S dense in E). Let E' be a complete metric space and $f: S \to E'$ a uniformly continuous function. Prove that f can be extended to a continuous function from E into E' in one and only one way, and that this extended function is also uniformly continuous

Let E, E' have metrics d, d' respectively

For any $x \in \complement S$ we can construct a sequence $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} x_n = x$ where $x_n \in S$, $x \in \complement S$

Since each element of $\complement S$ is a cluster point of S

Define the extension of f as $g: E \to E'$ where

$$g(x) = \begin{cases} f(x) & \text{if } x \in S \\ \lim_{n\to\infty} f(x_n), \text{ where } x_n \text{ is chosen such that } \lim_{n\to\infty} x_n = x, & \text{if } x \in \complement S \end{cases}$$

Proof that $\lim_{n\to\infty} f(x_n)$ exists:

Since $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence it is also a cauchy sequence

Since $f: S \to E'$ is uniformly continuous for any $\varepsilon > 0$ $\exists\ \delta > 0$ such that if

$$d(x_n, x_m) < \delta \text{ then } d'(f(x_n), f(x_m)) < \varepsilon \quad \forall x_n, x_m \in S$$

Since $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence $\exists$ a positive integer N such that

$$d(x_n, x_m) < \delta \quad \forall\ n, m > N$$

therefore $d'(f(x_n), f(x_m)) < \varepsilon \quad \forall\ n, m > N$

therefore $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence

Since E' is complete $\{f(x_n)\}_{n=1}^{\infty}$ converges

therefore $\lim_{n\to\infty} f(x_n)$ exists

Proof that g is uniformly continuous.

for any $x_1, x_2 \in E$ $\exists$ $\{x_n\}_{n=1}^{\infty}$, $\{x_n'\}_{n=1}^{\infty}$

such that $\lim_{n\to\infty} x_n = x_1$ and $\lim_{n\to\infty} x_n' = x_2$ and $x_n, x_n' \in S$ $\forall n$

$g(x_1) = \lim_{n\to\infty} f(x_n)$, $g(x_2) = \lim_{n\to\infty} f(x_n')$

$$d'(g(x_1), g(x_2)) \leq d'(g(x_1), f(x_n)) + d'(f(x_n), g(x_2))$$
$$d'(g(x_1), g(x_2)) \leq d'(g(x_1), f(x_n)) + d'(f(x_n), f(x_n')) + d'(f(x_n'), g(x_2))$$

$\exists$ a positive integer N such that

$d'(g(x_1), f(x_n)) < \frac{\varepsilon}{3}$ $\forall n > N$

$\exists$ a positive integer M such that

$d'(g(x_2), f(x_n')) < \frac{\varepsilon}{3}$ $\forall n > M$

Since f is uniformly continuous $\exists$ $\delta_0 > 0$ such that

$d'(f(x_n), f(x_n')) < \frac{\varepsilon}{3}$ whenever $d(x_n, x_n') < \delta_0$

fix $n > \max\{N, M\}$

$d'(g(x_1), g(x_2)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$ whenever $d(x_n, x_n') < \delta_0$

$d(x_n, x_n') \leq d(x_n, x_1) + d(x_1, x_n') \leq d(x_n, x_1) + d(x_1, x_2) + d(x_2, x_n') < \delta_0$

therefore $\exists$ $\delta > 0$ such that

$d'(g(x_1), g(x_2)) < \varepsilon$ whenever $d(x_1, x_2) < \delta$ $\forall x_1, x_2 \in E$

therefore g is uniformly continuous

Proof that g is formed in one and only one way:

Let $\{x_n\}_{n=1}^{\infty}$, $\{x_n'\}_{n=1}^{\infty}$ be two sequences

with $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_n' = x$

Since g is uniformly continuous

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} g(x_n') = g(x)$$
$$= \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(x_n') = g(x)$$

23. Use problem 22 to prove that if V is a finite dimensional vector space over $\mathbb{R}$ and $\|\cdot\|_1, \|\cdot\|_2$ are two norm functions on V, then $\exists$ positive real numbers m, M such that $m \leq \frac{\|x\|_1}{\|x\|_2} \leq M \quad \forall x \in V, x \neq 0$

$\underset{V_1}{\underset{\shortparallel}{(V, \|\cdot\|_1)}}$ and $\underset{V_2}{\underset{\shortparallel}{(V, \|\cdot\|_2)}}$ are normed vector spaces

if $f: V_2 \to V_1$ is a linear transformation

then f is continuous if and only if the set

☆ $\left\{ \frac{\|f(x)\|_2}{\|x\|_1} : x \in V_1, x \neq 0 \right\}$ is bounded

this is a result of problem 22.

Let $f(x) = x$ where $f: V_2 \to V_1$

pf f is linear: $f(x_1 + x_2) = x_1 + x_2 = f(x_1) + f(x_2) \quad \forall x_1, x_2 \in V_2$

$f(\alpha x_1) = \alpha x_1 = \alpha f(x_1) \quad \forall x_1 \in V_2, \alpha \in \mathbb{R}$

f is clearly continuous since it is the identity function

therefore the set $\left\{ \frac{\|f(x)\|_2}{\|x\|_1} : x \in V_1, x \neq 0 \right\}$ is bounded

$\left\{ \frac{\|x\|_2}{\|x\|_1} : x \in V_1, x \neq 0 \right\}$ is bounded

therefore there exists a positive real number M such that

$\frac{\|x\|_2}{\|x\|_1} \leq M \quad \begin{matrix} x \in V \\ x \neq 0 \end{matrix}$

Let $g(x) = x$ where $g: V_1 \to V_2$

g is a linear transformation and is continuous and therefore the set $\left\{ \frac{\|g(x)\|_1}{\|x\|_2} : x \in V_2, x \neq 0 \right\}$ is bounded

therefore the set $\left\{ \frac{\|x\|_1}{\|x\|_2} : x \in V_2, x \neq 0 \right\}$ is bounded

therefore there exists a positive real number M' such that

$\frac{\|x\|_1}{\|x\|_2} \leq M' \quad \begin{matrix} x \in V \\ x \neq 0 \end{matrix}$

$m = \frac{1}{M'} \leq \frac{\|x_2\|}{\|x\|_1} \leq M$ let $m = \frac{1}{M'} > 0$. $m \leq \frac{\|x_2\|}{\|x_1\|} \leq M$