Bernstein polynomials

Def: $B_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad k=0,1,2,\dots,n, \ \binom{n}{k} = \frac{n!}{k!(n-k)!}$

Clearly, $B_{n,k}(x) > 0$ on $(0,1)$.

a) $$\sum_{k=0}^{n} B_{n,k}(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = [x+(1-x)]^n = 1.$$

b) $$\sum_{k=0}^{n} k\, B_{n,k}(x) = nx$$

Pf: $$\sum_{k=0}^{n} k \frac{n!}{k!\,(n-k)!} x^k (1-x)^{n-k}$$

$$= nx \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} x^{k-1} (1-x)^{n-k}$$

Set $k = m+1$

$$= nx \sum_{m=0}^{n-1} \frac{(n-1)!}{m!\,(n-\cancel{1}-m)!} x^m (1-x)^{n-1-m}$$

$$= nx \sum_{m=0}^{n-1} \binom{n-1}{m} x^m (1-x)^{(n-1)-m} = nx. \quad \blacksquare$$

Likewise:

c) $$\sum_{k=0}^{n} k(k-1)\, B_{n,k}(x) = n(n-1)\, x^2$$

Pf: $$\sum_{k=0}^{n} k(k-1) \frac{n!}{k!\,(n-k)!} x^k (1-x)^{n-k} =$$

$$\sum_{k=2}^{n} \frac{n!}{(k-2)!\,(n-k)!}\, x^{k}(1-x)^{n-k} \qquad \text{Set } k=m+2$$

$$= \sum_{m=0}^{n-2} \frac{n!}{m!\,(n-2-m)!}\, x^{m+2}(1-x)^{n-2-m}$$

$$= n(n-1)x^2 \sum_{m=0}^{n-2} \binom{n-2}{m} x^{m}(1-x)^{(n-2)-m} = n(n-1)x^2 \qquad \blacksquare$$

<u>Lemma</u>

$$\sum_{k=0}^{n} (k-xn)^2 B_{n,k}(x) = nx(1-x)$$

<u>Pf</u>:

$$\sum_{k=0}^{n} (k^2 - 2kxn + x^2n^2)\, B_{n,k}(x)$$

$$= \sum_{k=0}^{n} \left(k(k-1) + k(-2xn+1) + x^2n^2\right) B_{n,k}(x)$$

$$= n(n-1)x^2 + (-2xn+1)nx + n^2x^2$$

$$= \cancel{n^2x^2} - nx^2 - \cancel{2x^2n^2} + nx + \cancel{n^2x^2} = nx(1-x). \qquad \blacksquare$$

The Weierstrass approximation theorem

Theorem

Let $f: [0,1] \longrightarrow \mathbb{R}$ be a continuous function. Consider the polynomial

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) B_{n,k}(x)$$

Then

$$\lim_{n\to\infty} B_n(f)(x) = f(x)$$

and the limit is uniform

Pf: Since f is continuous on $[0,1]$ and $[0,1]$ is compact, f is uniformly continuous. Pick $\varepsilon > 0$ arbitrary. There ex. $\delta > 0$, so that $|f(x) - f(y)| < \varepsilon/2$ for all $x, y \in [0,1]$ with $|x-y| < \delta$.

Write, for $x \in [0,1]$ fixed

$$B_n(f)(x) - f(x) = \sum_{k=0}^{n} \left[f\left(\tfrac{k}{n}\right) - f(x)\right] B_{n,k}(x).$$

This is true because of a).

For all k, with $\left|\frac{k}{n} - x\right| < \delta$ we have $\left|f\left(\frac{k}{n}\right) - f(x)\right| < \varepsilon/2$. For n large, so that $\frac{1}{n} < \delta$, there will always be such k's.

Hence we can estimate

$$|B_n(f)(x) - f(x)| \leq \sum_{k,\ |\frac{k}{n}-x| \leq \delta} \varepsilon \ B_{n,k}(x)$$

$$+ \sum_{k,\ |\frac{k}{n}-x| \geq \delta} |f(\tfrac{k}{n}) - f(x)|\, B_{n,k}(x).$$

Note that we used that $B_{n,k}(x) > 0$ on $(0,1)$.

Since f is uniformly continuous, it is bounded, i.e.,

$$|f(x)| \leq M \quad \text{for some} \quad M \in \mathbb{R},\ M > 0$$

and all $x \in [0,1]$. Hence, we continue and get

$$|B_n(f)(x) - f(x)| \leq \varepsilon/2 \sum_{k} B_{n,k}(x)$$

$$+ 2M \sum_{\substack{k \\ |\frac{k}{n}-x| \geq \delta}} B_{n,k}(x)$$

$$= \varepsilon/2 + 2M \sum_{\substack{k \\ |\frac{k}{n}-x| \geq \delta}} B_{n,k}(x).$$

Now we use our Lemma, i.e., we note that

$$\sum_{\substack{k \\ |\frac{k}{n}-x| \geq \delta}} B_{n,k}(x) = \sum_{\substack{k \\ |\frac{k}{n}-x| \geq \delta}} \frac{(\frac{k}{n}-x)^2}{(\frac{k}{n}-x)^2} B_{n,k}(x)$$

$$\leq \frac{1}{\delta^2} \sum_{\substack{k \\ |\frac{k}{n}-x| \geq \delta}} (\tfrac{k}{n}-x)^2 B_{n,k}(x) \leq \frac{1}{\delta^2} \sum_{k} (\tfrac{k}{n}-x)^2 B_{n,k}$$

$$= \frac{n x (1-x)}{n^2 \delta^2} = \frac{x(1-x)}{n \delta^2} \leq \frac{1}{4 n \delta^2}$$

since $x(1-x) \leq \frac{1}{4}$ on $[0,1]$.

Collecting the estimates we get

$$|B_n(f)(x) - f(x)| \leq \frac{\varepsilon}{2} + \frac{M}{2 n \delta^2}$$

Now we pick N so that

$$\frac{M}{N \delta^2} < \varepsilon$$

and hence we find that for any $\varepsilon > 0$ and all $n > N := \frac{M}{\varepsilon\, \delta(\varepsilon)^2}$

$$|B_n(f)(x) - f(x)| < \varepsilon.$$

Note that N depends on ε only and not on x. Thus the convergence is uniform. ▨