Final Exam for Anaysis I, Math 4317, December 13 , 2010, Allowed time is 2 hours and 50 minutes. This is a closed book test but a cheat sheet is allowed. Always state your reasoning otherwise credit will not be given. Try to be as concise and to the point as possible.

Name:

1: Consider the real line $\mathbb{R}$ with the metric $d(x,y) := |x-y|, x,y \in \mathbb{R}$ and consider the subsets of $\mathbb{R}$

$$A = (1,2)\ ,\ B = [1,2) \cup \{3\}\ ,\ C = [1,2]\ .$$

(4 points, no partial credit) Which ones are closed in $\mathbb{R}$?

(4 points, no partial credit) Which ones are open open in $\mathbb{R}$?

(4 points, no partial credit) Which ones are compact?

2: Consider the metric space $E = \{x \in \mathbb{R} : -1 < x < 1\}$ with the metric $d(x,y) = |x-y|, x,y \in \mathbb{R}$ and the subsets

$$A = (-1,0)\ ,\ B = (-1,0]\ ,\ C = [-1/2, 1/2]\ .$$

(4 points, no partial credit) Which ones are closed in E?

(4 points, no partial credit) Which ones are open open in E?

(4 points, no partial credit) Which ones are compact?

3: (16 points) Let $p_1, p_2, p_3, \dots$ be a sequence of points in a metric space E that converges to $p \in E$. Prove that the set $\{p, p_1, p_2, p_3, \dots\}$ is closed.

4: (12 points) Prove or disprove by finding a counterexample the following statement: If E is a metric space and $f : E \to E$ a continuous function then $f(S)$ is open for any open set $S \subset E$.

5: (12 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded and continuous function. Show that the function

$$F(x) = \int_0^x f(t)dt$$

a uniformly continuous function on $\mathbb{R}$.

6: (16 points) Show, by directly using the definition of the logarithm, that

$$\lim_{x \to 0} x^{\varepsilon} \log(x) = 0$$

for every $\varepsilon > 0$.

7: (20 points) Compute

$$\lim_{n\to\infty} \frac{1+2^k+3^k+\cdots n^k}{n^{k+1}} \ .$$

Here $k \in \mathbb{R}$ is a positive number.

8: Let C be a compact subset of $\mathbb{R}$. A function $f : C \to \mathbb{R}$ is upper semicontinuous if and only if for all $t \in \mathbb{R}$ the set $\{x \in C : f(x) \geq t\}$ is closed.

a) (5 points) Prove that any continuous function $f : C \to \mathbb{R}$ is upper semicontinuous.

b) (10 points) Prove that an upper semicontinuous function $f : C \to \mathbb{R}$ is bounded.

c) **Extra credit:** (15 points) Prove that an upper semicontinuous function $f : C \to \mathbb{R}$ attains its maximum value in the set C.

Extra Credit: (30 points) Prove that the series

$$\sum_{n,m=1}^{\infty} \frac{1}{(n+m)!}$$

converges and determine its value.