1 Integration of functions

In the following we consider the closed interval $[a,b] \subset \mathbb{R}$ and f a real valued, bounded function defined on $[a,b]$. Our goal is to give a definition of the Riemann integral and derive the fundamental theorem of calculus. I follow the great problem book of Polyá and Szgö "Aufgaben und Lehrsätze aus der Analysis I". I am sure that this book has been translated into English.

1.1 Partitions, upper sums and lower sums

A**partition $\mathcal{P}$ of the interval** $[a,b]$ is a collection of distinct points in

$$a = x_0 < x_1 < \cdots x_{n-1} < x_n = b \ .$$

Given two partitions $\mathcal{P}$ and $\mathcal{Q}$ we define the **refinement of $\mathcal{P}$ and $\mathcal{Q}$** to be

$$\mathcal{P} \cup \mathcal{Q} \ .$$

The upper sum

$$U_f(\mathcal{P}) = \sum_{j=1}^{n} \sup_{x_{j-1} \le x \le x_j} f(x)(x_j - x_{j-1})$$

and the lower sum

$$L_f(\mathcal{P}) = \sum_{j=1}^{n} \inf_{x_{j-1} \le x \le x_j} f(x)(x_j - x_{j-1}) \ .$$

Recall that

$$\sup_{x_{j-1} \le x \le x_j} f(x) = l.u.b.\{f(x) : x_{j-1} \le x \le x_j\}$$

and likewise

$$\inf_{x_{j-1} \le x \le x_j} f(x) = g.l.b.\{f(x) : x_{j-1} \le x \le x_j\} \ .$$

We have, obviously that

$$U_f(\mathcal{P}) \ge L_f(\mathcal{P})$$

and both sums are finite since the function is bounded.

1.1 LEMMA. *Let $\mathcal{P} \subset \mathcal{Q}$, i.e., $\mathcal{Q}$ is a refinement of $\mathcal{P}$. Then*

$$U_f(\mathcal{Q}) \le U_f(\mathcal{P})$$

and

$$L_f(\mathcal{Q}) \ge L_f(\mathcal{P}) \ .$$

Proof. Take two successive points $x_j > x_{j-1} \in \mathcal{P}$ for which there exists one or more points $y_1, \cdots, y_k$ in $\mathcal{Q}$ such that

$$x_{j-1} < y_1 < y_2 \cdots < y_k < x_j$$

Such a situation must exist since $\mathcal{Q}$ is a refinement of $\mathcal{P}$. Otherwise $\mathcal{P} = \mathcal{Q}$ and there is nothing to prove.

Now,

$$\sup_{x_{j-1} \le x \le x_j} f(x) \ge \max\{\sup_{x_{j-1} \le x \le y_1} f(x), \sup_{y_1 \le x \le y_2} f(x), \dots, \sup_{y_k \le x \le x_j} f(x)\}$$

and hence

$$\begin{aligned} & \sup_{x_{j-1} \le x \le x_j} f(x)(x_j - x_{j-1}) \\ \ge \quad & \sup_{x_{j-1} \le x \le y_1} f(x)(y_1 - x_{j-1}) + \sup_{y_1 \le x \le y_2} f(x)(y_2 - y_1) + \cdots + \sup_{y_k \le x \le x_j} f(x)(x_j - y_k) \ . \end{aligned}$$

This inequality proves that the first inequality of the lemma. The other follows in a similar fashion. □

1.2 COROLLARY. *Let $\mathcal{P}$ and $\mathcal{Q}$ be any two partitions. Then*

$$U_f(\mathcal{P}) \ge L_f(\mathcal{Q}) \ . \tag{1.1}$$

In particular

$$U_f = \text{g.l.b}\{U_f(\mathcal{P}) : \mathcal{P} \text{ is a partition}\}$$

and

$$L_f = \text{l.u.b}\{L_f(\mathcal{P}) : \mathcal{P} \text{ is a partition}\} \ ,$$

and

$$U_f \ge L_f \ .$$

We call the numbers U_f, L_f the upper respectively, lower limit.

Proof. Take the union $\mathcal{P} \cup \mathcal{Q}$ which is a refinement of both, $\mathcal{P}$ and $\mathcal{Q}$. By Lemma 1.1 we have that

$$U_f(\mathcal{P}) \ge U_f(\mathcal{P} \cup \mathcal{Q}) \ge L_f(\mathcal{P} \cup \mathcal{Q}) \ge L_f(\mathcal{Q}) \ .$$

The set $\{L_f(\mathcal{P}) : \mathcal{P} \text{ is a partition}\}$ is bounded above and the set $\{U_f(\mathcal{P}) : \mathcal{P} \text{ is a partition}\}$ is bounded below and therefore U_f and L_f are defined. To see that $U_f \ge L_f$ we assume on the contrary that $U_f < L_f$. This means that there is some $\varepsilon > 0$ so that $L_f > U_f + \varepsilon$. By the definition of L_f we can find a partition $\mathcal{P}$ so that $L_f(\mathcal{P}) > L_f - \varepsilon/2$ and a partition $\mathcal{Q}$ with $U_f(\mathcal{Q}) < U_f + \varepsilon/2$. This means that $L_f(\mathcal{P}) > U_f + \varepsilon/2 > U_f(\mathcal{Q})$, a contradiction to (1.1) □

1.3 DEFINITION. A function $f : [a, b] \to \mathbb{R}$ is **integrable** in the sense of Riemann, if it is bounded and if the upper limit equals the lower limit, i.e.,

$$U_f = L_f \ ,$$

and we denote this number by

$$\int_a^b f(x)dx \ .$$

1.4 Remark. *Thus, in order to decide whether a function is integrable we have to find a sequence of partitions $\mathcal{P}_n$ such that $U_f(\mathcal{P}_n) - L_f(\mathcal{P}_n)$ converges towards zero. Since*

$$0 \le U_f(\mathcal{P}_n) - U_f \le U_f(\mathcal{P}_n) - L_f(\mathcal{P}_n)$$

and since

$$0 \le L_f - L_f(\mathcal{P}_n) \le U_f(\mathcal{P}_n) - L_f(\mathcal{P}_n)$$

we learn that $U_f(\mathcal{P}_n)$ converges to U_f and $L_f(\mathcal{P}_n)$ converges to L_f. Thus $U_f = L_f$ and the function is integrable. There is of course great flexibility in finding such partitions.

1.2 Continuous functions and monotone functions are integrable

1.5 THEOREM. *Any bounded monotone function on the interval $[a, b]$ is integrable.*

Proof. We may assume that the function is monotone increasing. The proof for monotone decreasing functions follows by considering $-f$. All we have to do is to exhibit a sequence of partitions $\mathcal{P}_n$ so that $U_f(\mathcal{P}_n) - L_f(\mathcal{P}_n) \ge 0$ converges to zero. Pick

$$\mathcal{P}_n = \{a + \frac{k}{n}(b-a) : k = 1, \dots n\} \ .$$

Observe that

$$U_f(\mathcal{P}_n) - L_f(\mathcal{P}_n) = \sum_{j=1}^{n} \left[\sup_{x_{j-1} \le x \le x_j} f(x) - \inf_{x_{j-1} \le x \le x_j} f(x) \right] \frac{b-a}{n}$$

which equals

$$\sum_{j=1}^{n} [f(x_j) - f(x_{j-1})] \frac{b-a}{n} = \frac{(f(b) - f(a))(b-a)}{n}$$

which tends to zero as $n \to \infty$. □

For the next theorem the notion of **width of a partition** $\mathcal{P}$ which is defined as

$$\max\{x_j - x_{j-1} : 1 \le j < n\}$$

is useful.

1.6 THEOREM. *Any continuous function on the interval $[a, b]$ is integrable.*

Proof. Every continuous functions on a closed interval is uniformly continuous. Pick $\varepsilon > 0$. There exists $\delta >$ such that for all $x, y \in [a, b]$ with $|x - y| < \delta$ we have that

$$|f(x) - f(y)| < \frac{\varepsilon}{b-a}$$

Pick any partition $\mathcal{P}$ of width less than δ, e.g., the one before with

$$\frac{b-a}{n} < \delta \ .$$

Then

$$0 \leq U_f(\mathcal{P}) - L_f(\mathcal{P}) = \sum_{j=1}^{n} \left[\sup_{x_{j-1} \leq x \leq x_j} f(x) - \inf_{x_{j-1} \leq x \leq x_j} f(x) \right] (x_j - x_{j-1}) \ .$$

Further, since f is uniformly continuous on $[a, b]$ it is bounded and

$$\sup_{x_{j-1} \leq x \leq x_j} f(x) = f(x')$$

for some $x_{j-1} \leq x' \leq x_j$. Likewise

$$\inf_{x_{j-1} \leq x \leq x_j} f(x) = f(y')$$

for some $x_{j-1} \leq y' \leq x_j$. Since the width of the partition is less than δ we also have that $|x' - y'| < \delta$ and hence

$$0 \leq \left[\sup_{x_{j-1} \leq x \leq x_j} f(x) - \inf_{x_{j-1} \leq x \leq x_j} \right] f(x) = f(x') - f(y') < \frac{\varepsilon}{b-a} \ .$$

Thus

$$0 \leq U_f(\mathcal{P}) - L_f(\mathcal{P}) < \sum_{j=1}^{n} (x_j - x_{j-1}) \frac{\varepsilon}{b-a} = \varepsilon$$

Since ε is arbitrary, we have that $U_f = L_f$. $\square$

1.3 Some examples

Example 1: Consider the function $f(x)$ on $[0, 1]$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Pick any partition $\mathcal{P}$. Then

$$U_f(\mathcal{P}) = \sum_{j=1}^{n} \sup_{x_{j-1}\le x\le x_j} f(x)(x_j - x_{j-1}) = \sum_{j=1}^{n}(x_j - x_{j-1}) = 1$$

since every interval $[x_{j-1}, x_j]$ contains rational numbers. Likewise

$$L_f(\mathcal{P}) = 0$$

since every interval $[x_{j-1}, x_j]$ contains irrational numbers. Thus the upper limit $U_f = 1$ and the lower limit $L_f = 0$. This function is not integrable.

Example 2: Consider the functions $\frac{1}{x^2}$ on the interval $[a, b]$ with $a > 0$. Let $\mathcal{P}$ is any partition note that on the interval $[x_{j-1}, x_j]$ we have $\sup \frac{1}{x^2} = \frac{1}{x_{j-1}^2}$ and $\inf \frac{1}{x^2} = \frac{1}{x_j^2}$. Now

$$\frac{1}{x_{j-1}} - \frac{1}{x_j} = \frac{x_j - x_{j-1}}{x_j x_{j-1}}$$

and

$$\frac{1}{x_j^2}(x_j - x_{j-1}) \le \frac{x_j - x_{j-1}}{x_j x_{j-1}} \le \frac{1}{x_{j-1}^2}(x_j - x_{j-1})$$

we find that

$$L_f(\mathcal{P}) \le \sum_{j=1}^{n}\left(\frac{1}{x_{j-1}} - \frac{1}{x_j}\right) \le U_f(\mathcal{P}) \ .$$

But

$$\sum_{j=1}^{n}\left(\frac{1}{x_{j-1}} - \frac{1}{x_j}\right) = \frac{1}{a} - \frac{1}{b}$$

independent of the partition. Since $\frac{1}{x^2}$ is integrable on $[a, b]$ we find that

$$\int_a^b \frac{1}{x^2}dx = \frac{1}{a} - \frac{1}{b} \ .$$

Example 3: The function x^n, $n \in \mathbb{N}$, being continuous, is integrable on the interval $[a, b]$. We assume that $a > 0$. Once more choosing a partition we concentrate on the interval $[x_{j-1}, x_j]$ and note that

$$(x_j^{n+1} - x_{j-1}^{n+1}) = (x_j - x_{j-1})\sum_{k=0}^{n} x_j^k x_{j-1}^{n-k} \ .$$

Since $x_j > x_{j-1}$ we have that

$$(n+1)x_{j-1}^{n+1} < \sum_{k=0}^{n} x_j^k x_{j-1}^{n-k} < (n+1)x_j^{n+1} \ .$$

Hence, as before

$$L_f(\mathcal{P}) \leq \frac{\sum_{j=1}^{n}(x_j^{n+1} - x_{j-1}^{n+1})}{n+1} \leq U_f(\mathcal{P})$$

and once more we have a telescoping sum and obtain that for all partitions $\mathcal{P}$

$$L_f(\mathcal{P}) \leq \frac{b^{n+1} - a^{n+1}}{n+1} \leq U_f(\mathcal{P}) \ .$$

Hence

$$\int_a^b x^n dx = \frac{b^{n+1} - a^{n+1}}{n+1} \ .$$

An interesting example is given by the function $f(x) = \frac{1}{x}$ on $[a, b]$, where $a > 0$. Once more, this function is integrable and we try to compute the integral. Choose the sequence of partitions

$$\mathcal{P}_n = \{a \left(\frac{b}{a}\right)^{\frac{k}{n}} : k = 0, 1, \ldots, n\}$$

Now, compute

$$U_f(\mathcal{P}_n) = \sum_{j=1}^{n} \frac{1}{a\left(\frac{b}{a}\right)^{\frac{j-1}{n}}}(a\left(\frac{b}{a}\right)^{\frac{j}{n}} - a\left(\frac{b}{a}\right)^{\frac{j-1}{n}}) = n(\left(\frac{b}{a}\right)^{\frac{1}{n}} - 1)$$

and

$$L(\mathcal{P}_n) = \sum_{j=1}^{n} \frac{1}{a\left(\frac{b}{a}\right)^{\frac{j}{n}}}(a\left(\frac{b}{a}\right)^{\frac{j}{n}} - a\left(\frac{b}{a}\right)^{\frac{j-1}{n}}) = n(1 - \left(\frac{a}{b}\right)^{\frac{1}{n}})$$

Recall that

$$U_f(\mathcal{P}_n) \geq U_f \geq L_f \geq L_f(\mathcal{P}_n) \ .$$

Although we did not talk yet about the logarithm, it is easy to see that

$$\lim_{n\to\infty} n(\left(\frac{b}{a}\right)^{\frac{1}{n}} - 1) = \lim_{n\to\infty} = n(1 - \left(\frac{a}{b}\right)^{\frac{1}{n}}) = \log(\frac{b}{a}) \ .$$

Hence

$$\int_a^b \frac{1}{x} dx = \log(\frac{b}{a}) \ .$$

1.4 Linearity of the integral and Inequalities for integrals

1.7 THEOREM. *Let f and g be two integrable functions on the interval $[a, b]$. The $f + g$ is also integrable and*

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx \ .$$

Likewise, if $c \in \mathbb{R}$ is any constant the $cf(x)$ is integrable and

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx \ .$$

Proof. Pick any ε and choose partitions $\mathcal{P}$ and $\mathcal{Q}$ such that

$$\int_a^b f(x)dx - \varepsilon/2 < L_f(\mathcal{P}) \le U_f(\mathcal{P}) < \int_a^b f(x)dx + \varepsilon/2$$

and

$$\int_a^b g(x)dx - \varepsilon/2 < L_g(\mathcal{Q}) \le U_g(\mathcal{Q}) < \int_a^b g(x)dx + \varepsilon/2$$

Taking the refinement of the two partitions $\mathcal{R} = \mathcal{P} \cup \mathcal{Q}$ we know that

$$L_{f+g}(\mathcal{R}) \ge L_f(\mathcal{R}) + L_g(\mathcal{R}) \ ,$$

which follows from the fact that

$$\inf_S (f(x) + g(x)) \ge \inf_S f(x) + \inf_S g(x) \ .$$

Since

$$L_f(\mathcal{R}) + L_g(\mathcal{R}) \ge L_f(\mathcal{P}) + L_g(\mathcal{Q})$$

we have that

$$L_{f+g}(\mathcal{R}) > \int_a^b f(x)dx + \int_a^b g(x)dx - \varepsilon \ .$$

Similarly,

$$U_{f+g}(\mathcal{R}) \le U_f(\mathcal{R}) + U_g(\mathcal{R})$$

since

$$\sup_S (f(x) + g(x)) \le \sup_S f(x) + \sup_S g(x) \ .$$

Hence we have that

$$U_{f+g}(\mathcal{R}) \le U_f(\mathcal{P}) + U_g(\mathcal{Q}) < \int_a^b f(x)dx + \int_a^b g(x)dx + \varepsilon \ .$$

Thus

$$\int_a^b f(x)dx + \int_a^b g(x)dx - \varepsilon < L_{f+g}(\mathcal{R}) \le U_{f+g}(\mathcal{R}) < \int_a^b f(x)dx + \int_a^b g(x)dx + \varepsilon \ ,$$

which proves the additivity of the integral. The proof of the other statement is easy and is left as an excercise. □

Here is a little lemma concerning real functions defined on a set $S \subset \mathbb{R}$.

1.8 LEMMA. *Let f be a real valued function on a set $S \subset \mathbb{R}$. Then*

$$\sup_S f(x) - \inf_S f(x) \geq \sup_S |f(x)| - \inf_S |f(x)| \ .$$

Proof. We distinguish three cases.

a) $f(x) \geq 0$ for all $x \in S$. In this case, we have that $f(x) = |f(x)|$ and the inequality is an equality.

b) $f(x) \leq 0$ for all $x \in S$. In this case

$$\sup_S f(x) = -\inf_S(-f(x)) = -\inf_S |f(x)| \ .$$

Likewise

$$\inf_S f(x) = -\sup_S(-f(x)) = -\sup_S |f(x)|$$

and we have that

$$\sup_S f(x) - \inf_S f(x) = -\inf_S |f(x)| + \sup_S |f(x)|$$

and once more there is equality.

The interesting case is

c) $f(x)$ changes sign on S. Clearly

$$\sup f(x) = \sup\{f(x) : x \in S, f(x) > 0\}$$

and

$$\inf f(x) = \inf\{f(x) : x \in S, f(x) < 0\} \ ,$$

or

$$\inf f(x) = -\sup\{-f(x) : x \in S, -f(x) > 0\}$$

But,

$$\sup\{f(x) : x \in S, f(x) > 0\} + \sup\{-f(x) : x \in S, -f(x) > 0\} = \sup_S |f(x)|$$

since the sets where $f(x) > 0$ and the set where $f(x) < 0$ are disjoint. Hence

$$\sup_S f(x) - \inf_S f(x) = \sup_S |f(x)| \geq \sup_S |f(x)| - \inf_S |f(x)| \ .$$

and we are done.

$\square$

1.9 THEOREM. *Let f be an integrable function on $[a,b]$. Then its absolute value $|f|$ as well as its positive part defined by $f_+(x) = \max\{f(x), 0\}$ and its negative part defined by $f_-(x) = \max\{-f(x), 0\}$ are integrable.*

Proof. Consider the upper sum $U_f(\mathcal{P})$ and the lower sum $L_f(\mathcal{P})$ for the function $f(x)$, where $\mathcal{P}$ is a partition. Since

$$U_f(\mathcal{P}) - L_f(\mathcal{P}) = \sum_{j=1}^{n} \left[\sup_{x_{j-1} \leq x < x_j} f(x) - \inf_{x_{j-1} \leq x < x_j} f(x) \right] (x_j - x_{j-1}) \ .$$

Bu the above lemma we have

$$\sup_{x_{j-1} \leq x < x_j} f(x) - \inf_{x_{j-1} \leq x < x_j} f(x) \geq \sup_{x_{j-1} \geq x < x_j} |f(x)| - \inf_{x_{j-1} \leq x < x_j} |f(x)|$$

and hence

$$U_f(\mathcal{P}) - L_f(\mathcal{P}) \geq U_{|f|}(\mathcal{P}) - L_{|f|}(\mathcal{P})$$

and $|f|$ is integrable if f is integrable. Indeed, f integrable means that for any ε there exists a partition such that

$$\varepsilon > U_f(\mathcal{P}) - L_f(\mathcal{P})$$

and hence by the above

$$\varepsilon > U_{|f|}(\mathcal{P}) - L_{|f|}(\mathcal{P}) \geq 0 \ .$$

Since

$$f_+(x) = \frac{f(x) + |f(x)|}{2} \ , \ f_-(x) = \frac{-f(x) + |f(x)|}{2}$$

the integrability follows from the one of $|f|$ and the linearity of the integral. □

The following is immediate.

1.10 LEMMA. *Let f be an integrable function on $[a,b]$. Hence there exists a constant $M > 0$ such that $|f(x)| \leq M$ for all $\in [a,b]$. Then*

$$\left| \int_a^b f(x)dx \right| \leq M(b-a) \ .$$

1.5 Fundamental Theorem of Calculus

1.11 THEOREM. *Let f be a function that is integrable on $[a,b]$ and on $[b,c]$. Then f is integrable on $[a,c]$ and*

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx \ .$$

Proof. Pick any $\varepsilon > 0$ and let $\mathcal{P}$ be a partition of $[a,b]$ such that

$$\int_a^b f(x)dx - \varepsilon/2 < L_f(\mathcal{P}) \leq U_f(\mathcal{P}) < \int_a^b f(x)dx + \varepsilon/2$$

and $\mathcal{Q}$ be a partition of $[b,c]$ such that

$$\int_b^c f(x)dx - \varepsilon/2 < L_f(\mathcal{Q}) \leq U_f(\mathcal{Q}) < \int_b^c f(x)dx + \varepsilon/2$$

The union $\mathcal{R} = \mathcal{P} \cup \mathcal{Q}$, although not a refinement of $\mathcal{P}$ nor $\mathcal{Q}$ is a partition of the interval $[a,c]$. Further,

$$L_f(\mathcal{R}) = L_f(\mathcal{P}) + L_f(\mathcal{Q})$$

and

$$U_f(\mathcal{R}) = U_f(\mathcal{P}) + U_f(\mathcal{Q}) \ .$$

Hence,

$$\int_a^b f(x)dx + \int_b^c f(x)dx - \varepsilon < L_f(\mathcal{R}) \leq U_f(\mathcal{R}) < \int_a^b f(x)dx + \int_b^c f(x)dx + \varepsilon \ .$$

This implies that f is integrable on $[a,c]$ and its integral is the sum of the integrals on $a,b]$ and $[b,c]$. $\square$

It follows from the definition of the integral that for every function f,

$$\int_a^a f(x)dx = 0 \ .$$

We adopt the convention that

$$\int_a^b f(x)dx = -\int_b^a f(x)dx \ .$$

1.12 THEOREM. *Let $U \subset \mathbb{R}$ be an open interval and let $a \in U$ be any point. Let f be a continuous real valued function and define for any $x \in U$*

$$F(x) = \int_a^x f(t)dt \ .$$

The F is differentiable in U and

$$F'(x) = f(x)$$

all $x \in U$.

Proof. Fix and $x_0 \in U$. We have that

$$F(x) - F(x_0) = \int_{x_0}^{x} f(t)dt \ .$$

Hence

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| = \left| \frac{\int_{x_0}^{x} [f(t) - f(x_0)]dt}{x - x_0} \right| \ .$$

For any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon$$

for all x with $|x - x_0| < \delta$. Thus, by the Lemma above

$$\left| \int_{x_0}^{x} [f(t) - f(x_0)]dt \right| < \varepsilon |x - x_0|$$

for all x with $|x - x_0| < \delta$ and hence

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| < \varepsilon$$

for all x with $|x - x_0| < \delta$. Hence $F(x)$ is differentiable at x_0 and its derivative is $f(x_0)$. $\square$