Test 2 for Anaysis I, Math 4317, November 5 , 2010, Allowed time is 50 minutes. This is a closed book test. Always state your reasoning otherwise credit will not be given. Try to be as concise and to the point as possible.

Name:

1: a) (10 points) For which values of A is the function $f(x)$ on $[-1, 1]$ given by

$$f(x) = \begin{cases} (x+1)^2 & \text{if } 0 < x \leq 1 \ , \\ x + A & \text{if } -1 \leq x \leq 0. \end{cases}$$

continuous.

b) (5 points) Is the function you got in a) uniformly continuous?

2: (15 points) A function $f : \mathbb{R} \to \mathbb{R}$ is called uniformly Hölder-continuous of order $\alpha > 0$ if there exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}$

$$|f(x) - f(y)| \leq C|x - y|^\alpha \ .$$

Show that such a function is continuous.

3: (5 points) For $n = 1, 2, 3, \ldots$ consider the functions $f_n : \mathbb{R} \to \mathbb{R}$ given by

$$f_n(x) = \frac{1}{1 + (x - n)^2} \; .$$

a) Does this sequence of functions converge? If yes, what is the limiting function.

b) (5 points) Is the convergence uniform?

4: Let E, E' be two metric spaces and let $f : E \to E'$ be a continuous function.
a) (10 points) Prove that for every closed set $S \subset E'$. $f^{-1}(S)$ is closed in E.

Assume in addition that E is compact and that f is one-to-one and onto.
b) (20 points) Let q_n be a sequence in E' that converges to $q \in E'$. Prove that $p_n := f^{-1}(q_n)$ converges to $p := f^{-1}(q)$ in E. (Hint: Use the fact that if a sequence has the property that every convergent subsequence has the same limit, then the sequence converges.)

c) (5 points) What can you conclude from b) about the inverse function $f^{-1}(q)$, $q \in E'$?

5: True or false: (5 points each)

a) Every real function $f(x, y)$ on E^2 which, for every fixed x, is continuous as a function of y and which, for every fixed y, is continuous as a function of x, is continuous as a function from $E^2 \to \mathbb{R}$.

b) A continuous function $f : E \to E'$ where E, E' are metric space has the property that for any open set $S \subset E$, $f(S) \subset E'$ is also open.

c) Any convergent sequence of continuous functions defined on a compact metric space converges uniformly.

d) A continuous function defined on a compact metric space is uniformly continuous.

e) If $f : E \to E'$ is continuous and if E is compact, then $f(E)$ is also compact.

Additional credit: (15 points) Let $f : E \to E'$ be a function. Prove that the function is continuous if and only if for any subset $S \subset E'$

$$f^{-1}(S^\circ) = \left\{f^{-1}(S)\right\}^\circ$$

where S° is the open interior of S. Recall that

$$S^\circ = \cup_{U \subset S, open} U \ ,$$

i.e., the largest open set that is a subset of S.