

## 1. WEAK SEQUENCES ON EXISTENCE OF EIGENVALUES FOR COMPACT OPERATORS

Below is a short summary about weakly convergent sequences and its uses. Recall that a sequence  $f_n \in \mathcal{H}$  converges weakly to  $f \in \mathcal{H}$  if

$$\lim_{n \rightarrow \infty} (h, f_n) = (h, f)$$

for all elements  $h \in \mathcal{H}$ . As an example consider on orthonormal sequence  $e_n$  which converges weakly to 0. Clearly every strongly convergent sequence is weakly convergent. What the example just mentioned suggests is that there are “many” more sequences that converge weakly than there are that converge strongly.

Among the memorable facts are that any weakly convergent sequence is bounded, i.e., there exists a constant  $C$  such that  $\|f_n\| \leq C$  for all  $n = 1, 2, 3, \dots$ . The point about weak convergence is, however, the following theorem.

**Theorem 1.1. Weak sequential compactness** *Let  $\mathcal{H}$  be a Hilbert space. Then every bounded sequence  $f_n$  has a weakly convergent subsequence.*

*Proof.* A short sketch of the proof: let  $f_n$  be a bounded sequence. Then all the possible finite linear combination span a linear manifold  $\mathcal{M}$ . The closure, i.e., the intersection of all subspaces of  $\mathcal{H}$  that contain  $\mathcal{M}$  is the closure of  $\mathcal{M}$ . We denote this space by  $\mathcal{G}$ . It is a subspace of  $\mathcal{H}$ , it has a countable dense set of vectors (Why?) and hence forms a separable Hilbert space. We establish now the existence of a convergent subsequence in this Hilbert space. Denote by  $N \subset \mathcal{G}$  a countable dense set of vectors. Pick  $h_1 \in N$ . Since the sequence is bounded so is  $(h_1, f_n)$  and hence there exists a convergent subsequence  $(h_1, f_{n_1(k)})$ . Now pick  $h_2 \in N$  and a new subsequence  $n_2(k)$  of  $n_1(k)$  such that  $(h_2, f_{n_2(k)})$  converges. Continuing this way we have for any  $j = 1, 2, 3, \dots$

$$\lim_{k \rightarrow \infty} (h_j, f_{n_j(k)}) = c_j$$

and  $n_j(k)$  is a subsequence of  $n_{j-1}(k)$ . Now we consider the sequence

$$f_{n_k(k)}$$

and note that

$$\lim_{k \rightarrow \infty} (h_j, f_{n_k(k)}) = c_j .$$

Since  $f_{n_k(k)}$  is bounded we have that

$$\lim_{k \rightarrow \infty} (h, f_{n_k(k)})$$

converges for all  $h \in \mathcal{G}$  to some limit  $c(h)$ . If  $u \in \mathcal{H}$  we can write uniquely  $u = h + v$  where  $h \in \mathcal{G}$  and  $v \in \mathcal{G}^\perp$ . Since  $(v, f_n) = 0$  for all  $n = 1, 2, 3, \dots$  we find that

$$\lim_{k \rightarrow \infty} (u, f_{n_k(k)}) = \lim_{k \rightarrow \infty} (h, f_{n_k(k)}) = c(u) .$$

It is easy to see that  $u \rightarrow c(u)$  is linear and that  $|c(u)| \leq C\|u\|$ . Hence, by the Riesz representation theorem there exists  $f \in \mathcal{H}$  such that  $c(u) = (u, f)$  for all  $u \in \mathcal{H}$ . Hence,  $f_{n_k(k)}$  converges weakly to  $f$ . □

Let us turn now to compact operators. Recall  $A : \mathcal{H} \rightarrow \mathcal{H}$  is compact if it maps any bounded sequence into a sequence that has a *strongly* convergent subsequence. Alternatively, thanks to Theorem 1.1,  $A : \mathcal{H} \rightarrow \mathcal{H}$  is compact if and only if it maps weakly convergent sequences into

strongly convergent Also recall that  $f \in \mathcal{H}$  is an eigenvector of the linear operator  $A$  if there exists a number  $\lambda \in \mathbb{C}$  such that

$$Af = \lambda f .$$

**Theorem 1.2.** *Compact self-adjoint operators have eigenvalues let  $A : \mathcal{H} \rightarrow \mathcal{H}$  be a compact, self-adjoint operator. Then either  $\|A\|$  or  $-\|A\|$  is an eigenvalue of  $A$ .*

*Proof.* Here we recall that the sesquilinear form

$$\omega(f, g) := (Af, g)$$

satisfies

$$|\omega(f, g)| = |\omega(g, f)| ,$$

and hence

$$\sup_{\|f\|=1} |\omega(f, f)| = \sup_{\|f\|=1=\|g\|} |\omega(f, g)|$$

and hence

$$\sup_{\|f\|=1} |(Af, f)| = \|A\| .$$

We can always assume that

$$\sup_{\|f\|=1} (Af, f) = \|A\|$$

because otherwise we can replace  $A$  by  $-A$ . Our goal is to show that

$$\sup_{\|f\|=1} (Af, f)$$

is attained, i.e., there exists  $g \in \mathcal{H}$  with  $g \neq 0$  such that

$$Ag = \|A\|g .$$

By the definition of the sup there exists a maximizing sequence  $f_n$ , that is,  $\|f_n\| = 1, n = 1, 2, 3, \dots$  and

$$\lim_{n \rightarrow \infty} (Af_n, f_n) = \|A\| .$$

Since  $f_n$  is bounded there exists a weakly convergent subsequence which we again denote by  $f_n$  whose weak limit we call  $g$ . Now, you know from the exercises that

$$1 = \lim_{n \rightarrow \infty} \|f_n\| \geq \|g\| .$$

Since  $A$  is compact  $Af_n$  converges strongly to  $Ag$ . It is again an exercise to see that

$$\lim_{n \rightarrow \infty} (Af_n, f_n) = (Ag, g) .$$

In particular  $g \neq 0$  since

$$(Ag, g) = \|A\| .$$

Next we show that  $\|g\| = 1$ . Since  $0 < \|g\|$ ,

$$\|A\| \geq \left( A \frac{g}{\|g\|}, \frac{g}{\|g\|} \right) = \frac{(Ag, g)}{\|g\|^2} = \frac{\|A\|}{\|g\|^2}$$

and hence  $\|g\| \geq 1$  and therefore  $\|g\| = 1$ . Now we show that

$$Ag = \|A\|g .$$

Pick any  $v \in \mathcal{H}$  and consider the vector

$$f_t = \frac{g + tv}{\|g + tv\|} .$$

The number  $t \in \mathbb{C}$ . Now

$$\|A\| \geq (Af_t, f_t) = \frac{\|A\| + \bar{t}(Av, g) + t(Ag, v) + |t|^2(Av, v)}{1 + \bar{t}(v, g) + t(g, v) + |t|^2(v, v)}$$

The right side defines a real valued function  $F(t)$ . Since  $F(0) = \|A\|$ ,  $f$  has a maximum at  $t = 0$ . A short computation shows that for  $t$  real

$$0 = \frac{d}{dt}F(0) = (Av, g) + (Ag, v) - \|A\|[(v, g) + (g, v)]$$

and setting  $t = is$  we find that

$$0 = \frac{d}{ds}F(is)|_{s=0} = i[-(Av, g) + (Ag, v)] - \|A\|i[-(v, g) + (g, v)] .$$

Hence we have that

$$(Ag, v) = \|A\|(g, v)$$

for all  $v \in \mathcal{H}$ . Thus,  $Ag = \|A\|g$ . □

## 2. THE SPECTRAL THEOREM FOR COMPACT OPERATORS

We can now use Theorem 1.2 to prove the following theorem.

**Theorem 2.1. The Spectral theorem** *Let  $\mathcal{H}$  be a Hilbert space and  $A : \mathcal{H} \rightarrow \mathcal{H}$  a linear compact self-adjoint operator. Then there exists an orthonormal system  $\varphi_j, j = 1, 2, \dots$  and real numbers  $\lambda_j, j = 1, 2, \dots$  with  $\lim_{j \rightarrow \infty} \lambda_j = 0$  such that for all  $f \in \mathcal{H}$*

$$Af = \sum_{j=1}^{\infty} \lambda_j \varphi_j (\varphi_j, f) .$$

*If  $\mathcal{H}$  is separable, the system  $\varphi_j, j = 1, 2, \dots$  is a complete orthonormal system.*

*Proof.* By Theorem 1.2 there exists  $\varphi_1$ , normalized such that

$$A\varphi_1 = \pm \|A\|\varphi_1 .$$

Note that the minus sign applies if  $-\|A\|$  is the the eigenvalue with largest magnitude. Hence we set  $\lambda_1 = \pm \|A\|$ . The subspace  $M_1 = \{f \in \mathcal{H} : (\varphi_1, f) = 0\}$  is a closed subspace of  $\mathcal{H}$ . Moreover, for  $f \in M_1$ ,

$$(Af, \varphi_1) = (f, A\varphi_1) = \lambda_1 (f, \varphi_1) = 0$$

and hence  $A_1$  the restriction of  $A$  to the subspace  $M_1$  is a compact self-adjoint operator  $A_1 : M_1 \rightarrow M_1$ . Applying Theorem 1.2 again we obtain a normalized vector  $\varphi_2$  with

$$A_1\varphi_2 = \pm \|A_1\|\varphi_2 .$$

If  $\|A_1\| = 0$  then  $A_1$  is the zero operator and we are finished. If not we set  $\lambda_2 = \pm \|A_1\|$ . Now define the subspace  $M_2$  to be all those vectors in  $M_1$  that are perpendicular to the vector  $\varphi_2$ . The restriction of  $A_1$  to  $M_2$  defines a compact self-adjoint operator  $A_2$ . Continuing this way we find that either the procedure terminates or there is an infinite sequence pair  $\varphi_j, \lambda_j$  with

$$A\varphi_j = \lambda_j \varphi_j , j = 1, 2, 3, \dots .$$

The sequence  $\varphi_j$  is an orthonormal sequence and hence converges weakly to zero. Since  $A$  is compact  $A\varphi_j$  converges strongly to zero and hence  $\lim_{j=1} \lambda_j = 0$ . The vectors  $\varphi_j$  span a subspace  $R$  of  $\mathcal{H}$ . If  $f \perp R$  then  $Af$  must be the zero vector, otherwise we could repeat our procedure, contradicting the fact that  $\lim_{j=1} \lambda_j = 0$ . Hence  $R$  is an invariant subspace of  $A$  and  $A$  restricted to  $R$  is the zero operator. Thus for  $f \in \mathcal{H}$ ,

$$Af = \sum_{j=1}^{\infty} \lambda_j \varphi_j (\varphi_j, f) .$$

□