Test 2 for Calculus II, Math 1502 K1 - K6, October 2, 2013

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This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.

PRINT your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

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I: Calculate the limits:
a) (7 points) Find the Taylor series at $a = 0$ of the function

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \ .$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \ , e^{-x} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$$

Only the even terms survive in the sum.

$$\cosh(x) = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdot$$

or

$$\cosh(x) = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!} \ .$$

b) (7 points) Find the second order Taylor polynomial at $a = 0$, $P_2(x)$, of the function $f(x) = \sqrt{1+x}$.

$$f(x) = \sqrt{1+x} \ , \ f'(x) = \frac{1}{2\sqrt{1+x}} \ , \ f''(x) = -\frac{1}{4(1+x)^{3/2}}$$

$$P_2(x) = 1 + \frac{x}{2} - \frac{1}{2}\frac{x^2}{4} \ .$$

c) (11 points) Assume that f is a function with $|f^{(n)}(x)| \leq 1$ for all n and all real x. Find the least integer for which you can be sure that $P_n(1)$, the $n-th$ Taylor polynomial at $a = 0$ of f, approximates $f(1)$ within 0.001.

The remainder is given by

$$\frac{f^{(n+1)}(c)x^n}{(n+1)!}$$

which for $x = 1$ can be estimated as

$$\left|\frac{f^{(n+1)}(c)}{(n+1)!}\right| \leq \frac{1}{(n+1)!}$$

If $n = 3$ then $(3+1)! = 24$, if $n = 4$ then $(4+1)! = 120$ Since $4! < 100$ and $5! > 100$ we need $n = 4$.

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II: a) a) (8 points) Compute

$$\sum_{k=0}^{\infty} \frac{k}{3^k} \ .$$

We start with

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} k x^{k-1} \ , \ |x| < 1 \ .$$

Hence

$$\frac{x}{(1-x)^2} = \sum_{k=0}^{\infty} k x^k \ , \ |x| < 1 \ ,$$

from which it follows that

$$\sum_{k=0}^{\infty} \frac{k}{3^k} = \frac{1}{3} \frac{1}{(1-1/3)^2} = \frac{3}{4}$$

b) (8 points) Find the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(x+3)^n}{2^n n}$$

Root test yields

$$\lim_{n \to \infty} \frac{|x+3|}{2} \frac{1}{n^{1/n}} = \frac{|x+3|}{2} < 1$$

for convergence. Hence there is absolute convergence for all x with

$$|x+3| < 2 \ .$$

At $x = -1$ we have

$$\sum_{n=1}^{\infty} \frac{2^n}{2^n n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent. At $x = -5$ the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which is alternating and convergent.
c) (9 points) By integrating a power series compute

$$\int_0^1 e^{-x^8} dx$$

in terms of a series. How many terms in that series do you have to sum to obtain a value for this integral with an error of not more than $\frac{1}{150}$.

$$e^{-x^8} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n}}{n!}$$

so that

$$L = \int_0^1 e^{-x^8} dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(8n+1)n!}$$

This is an alternating series and

$$|L - s_n| < \frac{1}{(8(n+1)+1)(n+1)!}$$

For $n = 2$ we have that

$$\frac{1}{(8(n+1)+1)(n+1)!} = \frac{1}{25 \cdot 6} = \frac{1}{150} \, .$$

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III: Decide whether the following series converge or diverge. State the convergence test that you are using.
a) (8 points)

$$\sum_{n=1}^{\infty} \frac{n^n}{2^{(n^2)}}$$

Root test yields

$$\lim_{n\to\infty} \frac{n}{2^n} = 0$$

Hence it converges.
b) (8 points)

$$\sum_{k=1}^{\infty} \frac{(n!)^2}{(3n)!}$$

Ratio test:

$$\lim_{n\to\infty} \frac{(n+1)!(n+1)!}{(3n+3)!} \frac{(3n)!}{n!n!} = \lim_{n\to\infty} \frac{(n+1)^2}{(3n+3(3n+2)(3n+1)} = 0$$

Hence it converges.
c) (9 points) Which function does the power series

$$\sum_{n=1}^{\infty} \frac{(x+3)^n}{2^n n}$$

represent?
Recall that

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^n}{n}$$

Hence if we replace x by $\frac{x+3}{2}$ we get that

$$\sum_{n=1}^{\infty} \frac{(x+3)^n}{2^n n} = -\ln(1 - \frac{x+3}{2}) = -\ln(\frac{-1-x}{2}) \ .$$

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IV: No partial credit: Given the vectors $\vec{a} = \langle 1, 4, -1 \rangle$ and $\vec{b} = \langle -2, 1, 2 \rangle$.
a) (5 points) Compute $2\vec{a} + 3\vec{b}$

$$\langle -4, 11, 4 \rangle$$

b) (5 points) Find the vector $\vec{c}$ so that $\vec{a} + \vec{c} = \vec{b}$.

$$\vec{c} = \vec{b} - \vec{a} = \langle -3, -3, 3 \rangle$$

c) (5 points) Find the length of the vector $\vec{a}$.

$$\sqrt{18} = 3\sqrt{2}$$

d) (5 points) Find the angle between the vectors $\vec{a}$ and $\vec{b}$.

$$\vec{a} \cdot \vec{b} = -2 + 4 - 2 = 0$$

The angle is $\pi/2$.
e) (5 points) Find the projection of the vector $\langle 1, 2 \rangle$ onto the vector $\langle 1, 1 \rangle$.

$$\frac{\langle 1, 1 \rangle \cdot \langle 1, 2 \rangle}{|\langle 1, 1 \rangle|^2} \langle 1, 1 \rangle = \frac{3}{2} \langle 1, 1 \rangle \ .$$