Practice Test 3A for Calculus II, Math 1502, October 17, 2013

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PRINT Section:

PRINT Name of TA:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.

PRINT your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

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I: Consider the system of equations

$$\begin{aligned} x - 2y + az &= 2 \\ x + y + z &= 0 \\ 3y + z &= 2 \end{aligned}$$

a) (15 points) For which values of a, if any, does this system have a unique solution? Find the solution for any such value of a.

The augmented matrix is

$$\begin{bmatrix} 1 & -2 & a & 2 \\ 1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 2 \end{bmatrix}$$

which row reduces to
$$\begin{bmatrix} 1 & -2 & a & 2 \\ 0 & 3 & 1-a & -2 \\ 0 & 0 & a & 4 \end{bmatrix}$$

We have to make sure that the last column is not a pivotal column. Hence there is a unique solutions for all $a \neq 0$. This solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{8+2a}{3a} \\ -\frac{4-2a}{3a} \\ \frac{4}{a} \end{bmatrix}$$

b) (5 points) For which value of a, if any, does this system have infinitely many solutions? Find all the solutions for any such value of a.

For $a \neq 0$ there is not free variable in this system and hence the solution is always unique.

c) (5 points) For which value of a, if any, does this system have no solutions?

If $a = 0$, since that makes the last column a pivotal column.

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II: Are the following vectors linearly independent?

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ -2 \\ 1 \end{bmatrix}$$

For linear independence we have to check that

$$w \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + x \begin{bmatrix} -2 \\ 1 \\ 4 \\ -3 \end{bmatrix} + y \begin{bmatrix} -3 \\ -4 \\ 1 \\ 2 \end{bmatrix} + z \begin{bmatrix} -4 \\ 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

implies that $w = x = y = z = 0$.
We have to row reduce the augmented matrix

$$\begin{bmatrix} 1 & -2 & -3 & -4 & 0 \\ 2 & 1 & -4 & 3 & 0 \\ 3 & 4 & 1 & -2 & 0 \\ 4 & -3 & 2 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -3 & -4 & 0 \\ 0 & 5 & 2 & 11 & 0 \\ 0 & 10 & 10 & 10 & 0 \\ 0 & 5 & 14 & 17 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -3 & -4 & 0 \\ 0 & 5 & 2 & 11 & 0 \\ 0 & 0 & 6 & -12 & 0 \\ 0 & 0 & 12 & 6 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -3 & -4 & 0 \\ 0 & 5 & 2 & 11 & 0 \\ 0 & 0 & 6 & -12 & 0 \\ 0 & 0 & 0 & 30 & 0 \end{bmatrix}$$

There is no free variable in this system and hence the solution is unique. This unique solutions is the trivial solution and hence the vectors are linearly independent.

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III: Do the following vectors span R^3?

$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 0 \end{bmatrix}$$

To prove that these vectors span R^3 e have to show that given an arbitrary vector

$$\vec{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

there exist number x, y, z so that

$$x \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + z \begin{bmatrix} -1 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Hence we have to row reduce the augmented matrix

$$\begin{bmatrix} 1 & 2 & -1 & a \\ 4 & 1 & 10 & b \\ 2 & 3 & 0 & c \end{bmatrix}$$

which leads to

$$\begin{bmatrix} 1 & 2 & -1 & a \\ 0 & -7 & 14 & b-4a \\ 0 & -7 & 14 & 7c-14a \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 & -1 & a \\ 0 & -7 & 14 & b-4a \\ 0 & 0 & 0 & 7c-10a-b \end{bmatrix}$$

Hence, the vector $\vec{b}$ is in the span of the given vectors if and only if

$$7c - 10a - b = 0$$

These vectors therefore do not span R^3.

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IV: A linear transformation $T : R^3 \to R^2$ has the following properties

$$T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix} \quad , \quad T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix} \quad , \quad T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$$

Find the matrix associated with T.

We have to find $T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3)$. Now,

$$\vec{e}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix}$$

and hence

$$T(\vec{e}_1) = T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$$

Next

$$\vec{e}_3 = \begin{bmatrix}1\\0\\1\end{bmatrix} - \vec{e}_1$$

and hence

$$T(\vec{e}_3) = T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) - T(\vec{e}_1) = \begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$$

Finally,

$$\vec{e}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix} - \vec{e}_3$$

and hence

$$T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) - T(\vec{e}_3) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore, the associated matrix is given by

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

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V: Consider the linear transformation $T : R^3 \to R^3$ given by

$$T(\vec{e}_1) = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \; T(\vec{e}_2) = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}, \; T(\vec{e}_3) = \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix}$$

Is this transformation one-to one?

One to one means that $T(\vec{x}_1) = T(\vec{x}_2)$ implies that $\vec{x}_1 = \vec{x}_2$. Note that

$$T(\vec{x}_1) - T(\vec{x}_2) = T(\vec{x}_1 - \vec{x}_2)$$

and hence by setting $\vec{x} = \vec{x}_1 - \vec{x}_2$ this statement can be rephrased as $T(\vec{x}) = \vec{0}$ implies that $\vec{x} = 0$. In other words, we have to check that the equation

$$T(\vec{e}_1)x_1 + T(\vec{e}_2)x_2 + T(\vec{e}_3)x_3 = \vec{0}$$

has only the trivial solution.

The matrix associated with T is

$$\begin{bmatrix} 2 & -1 & 1 & 0 \\ 3 & 1 & 9 & 0 \\ 2 & -2 & -2 & 0 \end{bmatrix}$$

which row reduces to

$$\begin{bmatrix} 2 & -1 & 1 & 0 \\ 0 & 5 & 15 & 0 \\ 0 & -1 & -3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It has a free variable and hence it is not one-to-one. In fact all the solutions are given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$

where t is an arbitrary number.