Test 3 for Calculus II, Math 1502 K1 - K6, October 23, 2013

PRINT Name:

PRINT Section:

PRINT Name of TA:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.
PRINT your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

PRINT Name:

PRINT Section:

PRINT Name of TA:

I: Consider the system of equations

$$\begin{aligned} x + 3y - 5z &= -1 \\ 2x + y + 5z &= 8 \\ x + 2y - 2z &= b \end{aligned}$$

a) (10 points) Using row reduction reduce this system to *echelon form.*

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & -5 & -1 \\ 2 & 1 & 5 & 8 \\ 1 & 2 & -2 & b \end{bmatrix}$$

which upon row reduction yields

$$\begin{bmatrix} 1 & 3 & -5 & -1 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 0 & b-1 \end{bmatrix}$$

b) (2 points) For which values of b, if any, is the system consistent?

$b = 1$

c) (2 points) For which values of b, if any, is there a unique solution?

None

d) (6 points) For which values of b, if any, are there infinitely many solutions? Compute all the solutions for these cases.

$b = 1$ in which case the solutions are given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

PRINT Name:

PRINT Section:

PRINT Name of TA:

II: Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 2 \\ -13 \\ -5 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

a) (14 points) Determine all the vectors $\vec{b}$ that can be written as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Have to solve the system

$$x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = \vec{b}$$

The augmented matrix is

$$\begin{bmatrix} 2 & 1 & 2 & a \\ 1 & 4 & -13 & b \\ 1 & 2 & -5 & c \end{bmatrix}$$

which row reduces to

$$\begin{bmatrix} 1 & 4 & -3 & b \\ 0 & -2 & 8 & \frac{2a-4b}{7} \\ 0 & 0 & 0 & \frac{-2a-3b+7c}{7} \end{bmatrix}$$

Note that this reduction is not unique.

Precisely the vectors that satisfy

$$-2a - 3b + 7c = 0$$

can be written as a linear combination of $\vec{v}_1, \vec{v}_2\vec{v}_3$.

b) (6 points) Are the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ linearly independent?

No, because the third variable in the augmented matrix is free.

PRINT Name:

PRINT Section:

PRINT Name of TA:

III: Consider the matrix

$$B = \begin{bmatrix} 1 & 2 & 1 & 5 & 1 \\ 3 & 6 & 1 & 11 & 7 \\ 1 & 2 & 2 & 7 & -1 \end{bmatrix}$$

a) (10 points) Using row operations, bring this matrix to *reduced echelon form.*

$$\begin{bmatrix} 1 & 2 & 1 & 5 & 1 \\ 0 & 0 & -2 & -4 & 4 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix}$$

reduces to

$$\begin{bmatrix} 1 & 2 & 1 & 5 & 1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the reduced echelon form is

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b) (4 points) Indicate in the matrix B the pivotal positions.

The pivotal positions are the first position in the first row and the third position in the second row.

c) (6 points) The matrix B is the augmented matrix of the linear system

$$\begin{aligned} w + 2x + y + 5z &= 1 \\ 3w + 6x + y + 11z &= 7 \\ w + 2x + 2y + 7z &= -1 \end{aligned}$$

find all the solutions of this system.

We have that x and z are free variables and hence we may set $x = s, z = t$ and

$$y = -2 - 2t \ , w = -2s - 3t + 3$$

and hence

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

PRINT Name:

PRINT Section:

PRINT Name of TA:

IV: a) (10 points) Let $T : R^2 \to R^2$ be a linear transformation that maps the vector $\vec{e}_1$ to the vector $\vec{e}_1 + \vec{e}_2$ and the vector $\vec{e}_2$ to the vector $\vec{e}_1$. What is the matrix associated with T.

The matrix is given by

$$[\, T(\vec{e}_1) \quad T(\vec{e}_2) \,] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

b) (10 points) The linear transformations $Q : R^2 \to R^2$ is obtained by first performing the shear transformation

$$S\vec{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}$$

and then a rotation by $45°$. Find the matrix associated with Q.

The shear transformation maps the vectors $\vec{e}_1 \to \vec{e}_1$ and $\vec{e}_2 \to \vec{e}_1 + \vec{e}_2$. The rotation of $45°$ maps the vector $\vec{e}_1 \to \frac{1}{\sqrt{2}}[\vec{e}_1 + \vec{e}_2]$ and the vector $\vec{e}_2 \to \frac{1}{\sqrt{2}}[-\vec{e}_1 + \vec{e}_2]$. Hence

$$Q(\vec{e}_1) = \frac{1}{\sqrt{2}}[\vec{e}_1 + \vec{e}_2] \ , \ Q(\vec{e}_2) = \frac{1}{\sqrt{2}}[\vec{e}_1 + \vec{e}_2] + \frac{1}{\sqrt{2}}[-\vec{e}_1 + \vec{e}_2] = \sqrt{2}\vec{e}_2$$

Hence the matrix associated with Q is

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{2} \end{bmatrix}$$

PRINT Name:

PRINT Section:

PRINT Name of TA:

V: No partial credit: (5 points each) True or false:

a) A system of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ vectors in R^n with $p > n$ are always linearly dependent.

TRUE

b) If for a system of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in R^n every pair is linearly independent, then the whole system is linearly independent.

FALSE

c) For a linearly dependent system of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in R^n the vector $\vec{v}_1$ can always be expressed as a linear combination of the vectors $\vec{v}_2, \ldots, \vec{v}_p$.

FALSE

d) For a given system of linear equations, the echelon form is unique.

FALSE