Test 4 for Calculus II, Math 1502 K1 - K6, November 13, 2013

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This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write 1.414 . . .. Show your work, otherwise credit cannot be given.

PRINT your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

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I: a) (12 points) Find a basis for the Null Space of the matrix

$$A = \begin{bmatrix} 1 & 1 & -3 & -1 \\ 3 & 0 & 6 & 9 \\ 0 & 1 & -5 & -4 \end{bmatrix}$$

Row reduction leads to

$$\begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & 1 & -5 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced echelon form is convenient but not necessary:

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -5 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the Null Space of the matrix is given by all linear combinations of the form

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y - 3z \\ 5y + 4z \\ y \\ z \end{bmatrix}$$

$$= y \begin{bmatrix} -2 \\ 5 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

Hence, the basis of the Null Space is given by

$$\begin{bmatrix} -2 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

b) (2 points) What is the dimension of $Nul(A)$?

$$\dim Nul(A) = 2$$

c) (6 points) What is the dimension of $Col(A)$?

$$\dim Col(A) = 2$$

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II: a) (10 points) Given the vectors

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \ , \ \vec{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \ , \ \vec{c} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

compute the volume of the parallelepiped spanned by $\vec{a}, \vec{b}, \vec{c}$.

We compute the determinant of the matrix

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

which equals

$$2 - 3 \cdot 2 + [2 \cdot 1 - 2 \cdot 3] = -8$$

Hence the volume is 8.

b) (2 points) Are the vectors $\vec{a}, \vec{b}, \vec{c}$ linearly independent?

Yes the vectors are linearly independent since the determinant does not vanish.

c) (8 points) With the least amount of computation calculate the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} .$$

We swap the row containing the entries 1 three times so that the matrix has the form

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix} .$$

Now start the cofactor expansion with the last row and one gets that the determinant is 3 times the one determinant of the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

whose determinant is 1 times the determinant of the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

whose determinant is 1. Hence the determinant of the original matrix is -3.

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III: Given the matrices

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

a) (5 points) Show that the matrix $A = LU$ is invertible.

L is invertible because it has a pivot in every row and column. The same holds for U. The product of two invertible matrices is invertible.

b) (5 points) Compute the inverse of L.

Row reduction leads to

$$\begin{bmatrix} 1 & 0 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & -2 & 0 & 1 \end{bmatrix}$$

i.e.,

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

c) (5 points) Compute the inverse of U.

Once more row reduction leads to

$$U^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

d) Compute the inverse A^{-1}.

$$(LU)^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & 1 \end{bmatrix}$$

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IV: a) (8 points) Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{bmatrix} 7 & 8 \\ 2 & 1 \end{bmatrix}$$

The characteristic polynomial is

$$\lambda^2 - 8\lambda - 9 = (\lambda - 9)(\lambda + 1)$$

Since

$$A - 9I = \begin{bmatrix} -2 & 8 \\ 2 & -8 \end{bmatrix}$$

we find that all the eigenvectors for the eigenvalue 9 are a non-zero multiple of

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Likewise, since

$$A + I = \begin{bmatrix} 8 & 8 \\ 2 & 2 \end{bmatrix}$$

we have that all the eigenvectors for the eigenvalue -1 are a non-zero multiple of the vector

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

b) (4 points) Find the eigenvectors and eigenvalues for A^2.

The eigenvalues are 81 respectively 1 and the corresponding eigenvectors are

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -1 \end{bmatrix} .$$

c) (8 points) Find the eigenvalues and eigenvectors of the matrix

$$B = \begin{bmatrix} 13 & 16 \\ -9 & -11 \end{bmatrix}$$

The characteristic polynomial is

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$$

Since

$$B - I = \begin{bmatrix} 12 & 16 \\ -9 & -12 \end{bmatrix}$$

which reduces to

$$\begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$$

and we have that any eigenvector is a non-zero multiple of

$$\begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

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V: No partial credit: (5 points each) True or false (you do not have to give a reason):

a) Given a matrix A which is not invertible and a vector $\vec{b}$. Then $A\vec{x} = \vec{b}$ has no solution.

FALSE

b) An $n \times n$ matrix has n distinct eigenvalues. Then the eigenvectors are linearly independent.

TRUE

c) The collection of vectors $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that satisfy the equation $x+y+z=1$ form a subspace of $\mathcal{R}^3$.

FALSE

d) Let A be an $n \times n$ matrix. The matrix B obtain by adding a multiple of one row of A to another row of A does not change the eigenvalues.

FALSE