Homework IV, due Thursday April 10

I: (20 points) a) Let A be a linear operator on an n-dimensional space and assume that v is a cyclic vector. Show that $v, Av, \cdots, A^{n-1}v$ are linearly independent.

By Cayley's theorem,

$$\sum_{j=0}^{n} c_j A^j v = 0$$

where

$$p(x) = \sum_{j=0}^{n} c_j x^j$$

is $\det(A - xI)$, the characteristic polynomial of A. Since $c_n = \pm 1$ we have that

$$\pm A^n v = -\sum_{j=0}^{n-1} c_j A^j v$$

and hence $A^k v$ for all $k \geq n$ is a linear combination of $v, Av, \cdots, A^{n-1}v$. Since v is cyclic and hence $\{A^k v\}_{k \geq 0}$ spans the whole space, $v, Av, \cdots, A^{n-1}v$ must be a basis.

b) Show that a self adjoint operator on a finite dimensional space has a cyclic vector if and only if its eigenvalues have multiplicity one.

If the eigenvalues of A are distinct we may consider the vector

$$v = \sum_{j=1}^{n} v_j$$

where the vectors v_j are on eigenbasis. Now

$$A^k v = \sum_{j=1}^{n} \lambda_j^k v_j$$

The matrix with matrix elements $\lambda_j^k, 1 \leq j \leq n, 0 \leq k \leq n-1$ is a nonsingular matrix sine its determinant, the Vandermonde determinant is given by

$$\Pi_{j \neq k}(\lambda_j - \lambda_k) \neq 0 \ .$$

Hence the vectors $A^k v, k = 0, \ldots, n-1$ are linearly independent and hence v is a cyclic vector for A. Note that we did not use that A is self adjoint.

Now suppose that A has a degenerate eigenvalue λ. The characterisitic polynomial is of the form

$$p_A(x) = (x - \lambda)^a q(x)$$

where $a \geq 2$ and $q(x)$ has degree $n - a$. Since A is self adjoint the geometric and algebraic multiplicity of its eigenvalues are the same. Hence

$$r(A) = (A - \lambda I)q(A) = 0 \ .$$

(This is easily seen by diagonalizing A. In particular $r(A)v = 0$ and since the polynomial $r(x)$ has degree $< n$ the vectors $v, Av, \cdot, A^{n-1}v$ must be linearly dependent nonmatter what v Thus there is no cyclic vector for A.

Note that it is important that A is diagonalizable. E.g. the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has 0 has the only eigenvalue but has a cyclic vector.

II: (20 points) (taken from Reed and Simon) a) Let C be a symmetric operator, $A \subset C$ and assume that $\text{Ran}(A + iI) = \text{Ran}(C + iI)$. Show that $A = C$.

Let $f \in D(C)$. Then $(C + iI)f = (A + iI)g$ for some $g \in D(A)$, because $\text{Ran}(A + iI) = \text{Ran}(C + iI)$. Because $A \subset C$, $Ag = Cg$ and hence

$$(C + iI)f = (C + iI)g$$

or

$$C(f - g) + i(f - g) = 0 \ .$$

Thus, if $f - g \neq 0$, then it is an eigenvector of C with eigenvalue $-i$ and because C is symmetric this is impossible. Therefore $f = g$ and $f \in D(A)$.

b) Let A be a symmetric operator such that $\text{Ran}(A + iI) = \mathcal{H}$ but $\text{Ran}(A - iI) \neq \mathcal{H}$. Show that A does not have a self adjoint extension.

If B is a self adjoint extension of A, then $\text{Ran}(B + iI) = \mathcal{H} = \text{Ran}(A + iI)$. Because $A \subset B$ and B is symmetric, it follows from part a) that $A = B$. Thus, A is self adjoint and therefore $\text{Ran}(A - iI) = \mathcal{H}$ which is a contradiction.

III: (30 points) Let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator and assume $U - I$ is injective, i.e., 1 is not an eigenvalue of U.

a) Prove that $\text{Ran}(U - I)$ is dense in $\mathcal{H}$.

If there exist $g \neq 0$ with

$$\langle (U - I)f, g \rangle = 0$$

for all $f \in \mathcal{H}$ then

$$\langle f, (U^{-1} - I)g \rangle = 0$$

for all $f \in \mathcal{H}$ and hence $U^{-1}g = g$ or $Ug = g$ which is a contradiction.

b) Consider the mean

$$V_N = \frac{1}{N} \sum_{n=0}^{N-1} U^n \ .$$

Prove that for any $f \in \mathcal{H}$

$$\lim_{N \to \infty} \|V_N f\| = 0 \ .$$

Since $\mathrm{Ran}(U - I)$ is dense, for any ε we can find g so that $\|f - (U - I)g\| < \varepsilon$. Now

$$V_N(U - I)g = \frac{1}{N} \left[\sum_{n=1}^{N+1} U^n g - \sum_{n=0}^{N} U^n g \right] = \frac{1}{N}(U^{N+1} g - g)$$

which tends to zero in norm as $N \to \infty$. Next, V_N is uniformly bounded because

$$\|V_N f\| \leq \frac{1}{N} \sum_{n=0}^{N-1} \|U^n f\| \leq \|f\| \ ,$$

so that

$$\|V_N f\| \leq \|V_N[f - (U - I)g]\| + \|V_N(U - I)g\| \leq \varepsilon + \|V_N(U - I)g\| \ .$$

Hence

$$\lim_{N \to \infty} \|V_N f\| < \varepsilon$$

for any ε which proves the claim.

IV: (20 points) Let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator and form the mean

$$V_N = \frac{1}{N} \sum_{n=0}^{N-1} U^n \ .$$

Show that for any $f \in \mathcal{H}$

$$\lim_{N \to \infty} \|V_N f - P f\| = 0$$

where P is the projection onto the eigenspace of U with eigenvalue 1.

Note that

$$\overline{\mathrm{Ran}(U - I)} \oplus \mathrm{Ker}(U^* - I) = \mathcal{H} \ .$$

If $v \in \mathrm{Ker}(U^* - I)$then $V_N v = v$. If $v \in \mathrm{Ran}(U - I)$then $V_N v \to 0$ as $N \to \infty$.