BANACH ALGEBRAS

1. Basic definitions, invertibility

There are a number of books and we follow Kolmogorov-Fomin, Rudin, Neumark, Milman and Arveson. I also will be using some notes by Eric Carlen. In this section we shall talk about algebras over the complex numbers $\mathbb{C}$. An algebra is a vector space together with a multiplication satisfying

$$x(yz) = (xy)z$$

$$(x+y)z = xz + yz \ , \ z(x+y) = zx + zy$$

and for $\alpha \in \mathbb{C}$,

$$\alpha(xy) = (\alpha x)y = x(\alpha y)$$

Definition 1.1. *A* **Banach Algebra** $\mathcal{A}$ *is an Algebra that is a Banach space with a norm that satisfies*

$$\|xy\| \le \|x\|\|y\| \ ,$$

and there exists a unit element $e \in \mathcal{A}$ *such that* $ex = xe = x$, $\|e\| = 1$.

Examples:

a) The space of continuous functions $C(K)$ where K is compact and with the supremums norm.

b) The space of all complex values functions in $L^1(\mathbb{S})$ with the product

$$f \star g(x) = \int_{\mathbb{S}} f(x-y)g(y)dy \ .$$

c) The space of all bounded linear operators on a Hilbert space.

d) The space of all complex valued functions $f \in L^1(\mathbb{R})$ together with the product

$$f \star g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy \ .$$

This is not a Banach Algebra in the sense given above, because there is no unit element.

If $\mathcal{A}$ does not have a unit element then we can consider the collection of pairs

$$(\alpha, x) \ , \ \alpha \in \mathbb{C} \ , x \in \mathcal{A}$$

with the multiplication

$$(\alpha, x) \cdot (\beta, y) = (\alpha\beta, xy + \alpha y + \beta x)$$

and norm

$$\|(\alpha, x)\| = |\alpha| + \|x\|$$

contains the unit element $(1, 0)$. This is a Banach Algebra with unit element. Hence, it is no restriction of generality to consider Banach Algebras that contain a unit element.

Proposition 1.2. *In a Banach Algebra multiplication is continuous.*

Proof. Suppose that $x_n \to x$ and $y_n \to y$. Then

$$x_n y_n - xy = (x_n - x)y_n + x(y_n - y)$$

and since $\|y_n\|$ is bounded

$$\|x_n y_n - xy\| \le \|(x_n - x)\| \|y_n\| + \|x\| \|(y_n - y)\| \to 0 \ .$$

□

Definition 1.3. *An element $x \in \mathcal{A}$ is invertible if the exists an element $x^{-1} \in \mathcal{A}$ such that*

$$xx^{-1} = x^{-1}x = e \ .$$

Inverses are unique because $ax = xa = e, bx = xb = e$ then

$$a = ae = a(xb) = (ax)b = eb = b \ .$$

Proposition 1.4 (Neumann series.)**.** *Let $x \in \mathcal{A}$ and $\|x\| < 1$. Then $(e - x)$ has an inverse.*

Proof. Use the series

$$s_N = \sum_{n=0}^{N} x^n = e + x + x^2 + x^3 + \cdots + x^N$$

which is a Cauchy sequence and hence converges to some element s. Moreover

$$(e - x)s_N = e - x^{N+1} \to e$$

as $N \to \infty$. The same is true for $s_N(e - x)$ and by the continuity of multiplication we have that

$$s(x - e) = (x - e)s = e \ .$$

□

Definition 1.5. *Let $x \in \mathcal{A}$ be given. The set of all $\lambda \in \mathbb{C}$ such that $(\lambda e - x)$ is not invertible is called the spectrum of x and denoted by $\sigma(x)$. The spectral radius $\rho(x)$ is defined by*

$$\rho(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$$

Theorem 1.6. *Let $x \in \mathcal{A}$ and $p(\mu), \mu \in \mathbb{C}$ a polynomial with complex coefficients. Then we have for the spectrum of $p(x) \in \mathcal{A}$*

$$\sigma(p(x)) = p(\sigma(x)) \ .$$

Proof. We assume that the degree of p is N. Let λ be a point in $\sigma(p(x))$, i.e., the element $(\lambda e - p(x))$ has no inverse. The polynomial $\lambda - p(\mu)$ can be decomposed into linear factors

$$\lambda - p(\mu) = \Pi_{k=1}^{N}(\kappa_j - \mu) \tag{1}$$

where κ_j are the roots counted with multiplicity. Hence

$$\lambda e - p(x) = \Pi_{k=1}^{N}(\kappa_j e - x)$$

and since his element has no inverse at least one of the factor has no inverse. Hence $\kappa_j \in \sigma(x)$ for some j and since $p(\kappa_j) = \lambda$ we have that $\lambda \in p(\sigma(x))$. Conversely, assume that $\lambda \in p(\sigma(x))$, i.e., $p(\nu) = \lambda$ for some $\nu \in \sigma(x)$. Using (1) $\lambda - p(\nu) = \Pi_{k=1}^{N}(\kappa_j - \nu) = 0$ and hence $\nu = \kappa_j$ for some j. This means that the element $(\kappa_j e - x)$ is not invertible and hence $\kappa_j \in \sigma(x)$. Hence $\lambda e - p(x)$ is not invertible and $\lambda \in \sigma(p(x))$. □

By the Neumann series we know that $\rho(x) \leq \|x\|$ and hence we have that

$$\|x^n\| \geq \rho(x^n) = \sup\{|\lambda|^n : \lambda \in \sigma(x)\} = \rho(x)^n$$

which proves

Corollary 1.7. *We have that*

$$\rho(x) \leq \liminf_{n\to\infty} \|x^n\|^{1/n} \ .$$

The following Lemma is sometimes useful. It allows to replace lim inf by lim in the above formula.

Lemma 1.8. *Let $a_n, n = 1, 2, \dots$ be a sequence of positive real numbers such that $a_{n+m} \leq a_n a_m$. Then $\lim_{n\to\infty} a_n^{1/n}$ exists.*

Proof. Clearly

$$a_n \leq a_1^n$$

and hence $a_n^{1/n}$ is bounded. Fix $k \in \mathbb{N}$. Any number n can be written

$$n = l_n k + r_n$$

where $0 \leq r_n < k$. Hence

$$a_n \leq a_{l_n k} a_{r_n} \leq a_k^{l_n} a_{r_n}$$

so that

$$a_n^{1/n} \leq a_k^{l_n/n} a_{r_n}^{1/n} \ .$$

As $n \to \infty$, $l_n/n \to 1/k$ and $a_{r_n}^{1/n} \to 1$ so that we have

$$\limsup_{n\to\infty} a_n^{1/n} \leq a_k^{1/k}$$

for any k and hence

$$\limsup_{n\to\infty} a_n^{1/n} \leq \liminf_{k\to\infty} a_k^{1/k}$$

and the limit exists. □

Theorem 1.9. *Let $x \in \mathcal{A}$. Then the set $\sigma(x)$ is non-empty and compact. Moreover, the limit $\lim_{n\to\infty} \|x^n\|^{1/n}$ exists and equals to the spectral radius $\rho(x)$.*

Proof. Picking $\lambda > \|x\|$ we have that

$$(\lambda e - x) = \lambda(e - \frac{x}{\lambda})$$

and since $\|\frac{x}{\lambda}\| < 1$, $(\lambda e - x)$ is invertible. Hence $\sigma(x)$ is a bounded set. For $\lambda, \mu \notin \sigma(x)$ we have that

$$(\lambda e - x)^{-1} - (\mu e - x)^{-1} = (\mu - \lambda)(\lambda e - x)^{-1}(\mu e - x)^{-1}$$

or

$$(\mu e - x)^{-1} = (\lambda e - x)^{-1}[e - (\mu - \lambda)(\mu e - x)^{-1}] \ .$$

and the second factor has an inverse for $|\mu - \lambda|$ small enough which shows that whenever $\mu \notin \sigma(x)$ then $\lambda \notin \sigma(x)$ for $|\mu - \lambda|$ small enough. It follows that the complement of $\sigma(x)$ is open and hence $\sigma(x)$ is closed and bounded and hence compact. Pick any bounded linear function $f : \mathcal{A} \to \mathbb{C}$. We note that for λ, μ not in $\sigma(x)$ we have that

$$f((\lambda e - x)^{-1}) - f((\mu e - x)^{-1}) = (\mu - \lambda) f((\lambda e - x)^{-1}(\mu e - x)^{-1})$$

so that as $\mu \to \lambda$ we have that

$$\frac{d}{d\lambda} f((\lambda e - x)^{-1}) = -f((\lambda e - x)^{-2}) \ .$$

Hence, the function

$$\lambda \to f((\lambda e - x)^{-1})$$

is analytic on the complement of $\sigma(x)$. For $|\lambda| > \|x\|$ we find that

$$|f((\lambda e - x)^{-1})| \le \|f\| \|(\lambda e - x)^{-1}\| \le \frac{\|f\|}{|\lambda|} \|(e - \frac{x}{\lambda})^{-1}\| \le \frac{C}{|\lambda|} \to 0$$

as $\lambda| \to \infty$. Suppose that $\sigma(x) = \emptyset$. Then $f((\lambda e - x)^{-1})$ is an entire function that vanishes at infinity and hence by Liouville's theorem it is zero. Hence

$$f(x^{-1}) = 0$$

for every $f \in \mathcal{A}^*$ which yields that $x^{-1} = 0$ which is a contradiction.

It remains to show that $\rho(x) \ge \liminf_{n\to\infty} \|x^n\|^{1/n}$. For $|\lambda| > \|x\|$ we have that

$$f((\lambda e - x)^{-1}) = \sum_{n=0}^{\infty} \frac{f(x^n)}{\lambda^{n+1}} \ .$$

This representation also holds for all $|\lambda| > \rho(x)$ because the radius of convergence is given by the distance to the nearest singularity. Hence

$$\sup_n \frac{|f(x^n)|}{|\lambda|^{n+1}} < \infty$$

By the principle of uniform boundedness we find that

$$\frac{\|x^n\|}{|\lambda|^{n+1}} \le C(\lambda)$$

and hence

$$\limsup_{n\to\infty} \|x^n\|^{1/n} \le |\lambda|$$

for all λ with $|\lambda| > \rho(x)$. Hence

$$\rho(x) \ge \limsup_{n\to\infty} \|x^n\|^{1/n} \ .$$

□

2. Ideals, Gelfand transform

Another important concept is the **ideal**. We shall assume that $\mathcal{A}$ is a commutative Banach Algebra.

Definition 2.1. *A subspace $\mathcal{I}$ of a commutative Banach Algebra $\mathcal{A}$ is called an ideal if for any $x \in \mathcal{I}$ it follows that $xy \in \mathcal{I}$ for all $y \in \mathcal{A}$.*

There are two trivial ideals, the one consisting of the zero element and the one consisting of $\mathcal{A}$ itself and we shall assume that all the ideals under consideration are non-trivial. Note, any ideal that contains the unit element e is trivial. Hence an ideal consist only of non-invertible elements. An ideal is **maximal** if it is not contained in any other non-trivial ideal. An example is furnished by the Banach Algebra $C(K)$ of complex valued continuous function on a compact set K with the supremum norm.

Theorem 2.2. *Any maximal ideal $\mathcal{I} \subset C(K)$ consists of all the functions in $C(K)$ that vanish at some fixed but arbitrary point.*

Proof. First we show that the set $\mathcal{I}_{x_0}$ consisting of all the functions in $C(K)$ that vanish at some point $x_0 \in K$ form a maximal ideal. That they form an ideal is clear. Suppose that $\mathcal{I}_{x_0}$ is not maximal. Hence it is a proper subset of an ideal J. Let $f_0 \in J \setminus \mathcal{I}_{x_0}$. Then $f_0(x_0) \neq 0$ and for an arbitrary function $f \in C(K)$ we have that

$$f(x) = g(x) + \frac{f(x_0)}{f_0(x_0)} f_0(x)$$

where g vanishes at x_0. Hence $g \in \mathcal{I}_{x_0} \subset J$ and it follows that $J = C(K)$, i.e., trivial. Now, suppose that $\mathcal{I}$ is any maximal ideal. we have to show that there exists a point x_0 so that all functions that belong to $\mathcal{I}$ vanish at that point. Suppose there is no such point. Hence for every point $x_0 \in K$ we find a continuous function f_{x_0} that does not vanish at x_0. Since it is continuous there exists an open ball centered at x_0 such that f_{x_0} does not vanish on this ball. The collection of such balls form an open cover of K and hence, since K is compact, there exists a finite sub-cover, $B(x_1), \dots, B(x_N)$ and functions $f_{x_1}, \dots, f_{x_N}$ so that f_{x_i} does not vanish in $B(x_i)$. The function

$$g(x) = \sum_{i=1}^{N} |f_{x_i}(x)|^2$$

belongs to the ideal. $g(x) > 0$ on K and hence $1/g(x)$ is continuous. Thus

$$1 = \frac{1}{g(x)} g(x)$$

also belongs to the ideal and hence the ideal is trivial. □

Note that we have obtained a one to one correspondence between the points in K and the maximal ideals. Thus, we may consider the continuous function on K as functions from the set of maximal ideals into the complex numbers. This point of view is fruitful and should be kept in mind in what follows since it will allows us to realize commutative Banach Algebras as functions on a compact space.

Lemma 2.3. *Any non-trivial ideal is a subset of a maximal ideal.*

Proof. Denote the ideal by $\mathcal{I}$. A partial order among ideals containing $\mathcal{I}$ is established through inclusion. Consider any chain of such non-trivial ideals $\mathcal{I}_\alpha$, i.e. for any $\alpha \neq \beta$ either $\mathcal{I}_\alpha \subset \mathcal{I}_\beta$ or $\mathcal{I}_\beta \subset \mathcal{I}_\alpha$. We claim that $\mathcal{U} = \cup_\alpha \mathcal{I}_\alpha$ contains $\mathcal{I}$, is an ideal and hence an upper bound. Clearly, $\mathcal{U}$ is a subspace since all $\mathcal{I}_\beta$ are subspaces. Any $x \in \mathcal{U}$ is in some $\mathcal{I}_\alpha$ and if $y \in \mathcal{A}$ is any element we have that $xy \in \mathcal{I}_\alpha$ und thus in $\mathcal{U}$. Since e is not in any of the $\mathcal{I}_\beta$ it is not in $\mathcal{U}$ and hence $\mathcal{U}$ is nontrivial. That $\mathcal{I} \subset \mathcal{U}$ is evident. By Zorn's lemma there exists a maximal element $\mathcal{M}$, i.e., $\mathcal{M}$ is an ideal such that whenver $\mathcal{V}$ is an ideal that contains $\mathcal{I}$ and $\mathcal{M}$ then $\mathcal{V} = \mathcal{M}$. Hence $\mathcal{M}$ is a maximal ideal. □

Corollary 2.4. *The closure of an ideal is an ideal. In particular any maximal ideal is closed.*

Proof. This follows from the continuity of multiplication and the fact that the closure of a non-trivial ideal is nontrivial. □

Corollary 2.5. *An element x in a Banach algebra is invertible if and only if it is not a member of a maximal ideal. In particular, a Banach Algebra that has no non-trivial maximal ideals is a field.*

Proof. If x is invertible and member of an ideal, then this ideal is $\mathcal{A}$ and hence trivial. Conversely, assume that x is not invertible. We have to show that it is a member of a maximal ideal. Consider the space $\mathcal{I} = \{y \in \mathcal{A} : y = xz, z \in \mathcal{A}\}$. This space is linear. Since $\mathcal{A}$ is commutative, it is an ideal. Moreover, it is not trivial, since it does not contain e. Otherwise x would be invertible. Hence $\mathcal{I}$ is an ideal and hence contained in a non-trivial maximal ideal. □

Let $\mathcal{I}$ be a closed ideal in the Banach Algebra $\mathcal{A}$. The quotient space $\mathcal{A}/\mathcal{I}$ is again a Banach Space. Recall that the norm is given by

$$\|[x]\| = \inf_{z \in \mathcal{I}} \|x - z\| \ .$$

Next we define the multiplication. Let $[x], [y] \in \mathcal{A}/\mathcal{I}$ where x, y are representatives of the respective equivalence classes. Then we define

$$[x][y] = [xy] \ .$$

With this multiplication $[x], [y] \in \mathcal{A}/\mathcal{I}$ is a Banach Algebra, the factor algebra. We leave the proof of this as an exercise for the reader.

Lemma 2.6. *An ideal $\mathcal{I}$ is a proper subset of a nontrivial ideal if and only if its factor algebra $\mathcal{A}/\mathcal{I}$ has nontrivial ideals.*

Proof. Let $\mathcal{I} \subset \mathcal{J} \subset \mathcal{A}$, $\mathcal{J} \neq \mathcal{A}$. In the equivalence class $[x] = x + \mathcal{I}$ consider the subclass $[x'] = x' + \mathcal{J}$. It is easy to check that one obtains an ideal in the factor algebra. The proof of the converse is analogous. □

Corollary 2.7. *An ideal $\mathcal{I} \in \mathcal{A}$ is maximal if and only if $\mathcal{A}/\mathcal{I}$ is a field.*

Proof. This follows from the previous lemma and corollary. □

Theorem 2.8 (Gelfand-Mazur)**.** *If a Banach algebra is a field, i.e., all non-zero elements are invertible, then it is isometrically isomorphic to $\mathbb{C}$.*

Proof. Let $\lambda \in C$ be a number so that $(\lambda e - x)$ is not invertible. Such a number exists, since $\sigma(x)$ is not empty. Hence $\lambda e - x = 0$ and $x = \lambda e$. Hence, for every x there exists a unique $\lambda \in \mathbb{C}$ so that $x = \lambda e$ and conversely for $\lambda \in \mathbb{C}$ we find $x = \lambda e$. Thus the map $x \to \lambda$ is an isomorphism. Since $\|x\| = |\lambda|\|e\| = |\lambda|$, this isomorphism is an isometry. □

Corollary 2.9. *If $\mathcal{I}$ is a maximal ideal then $\mathcal{A}/\mathcal{I}$ is isometrically isomorph to $\mathbb{C}$.*

Corollary 2.10. *A closed ideal $\mathcal{M}$ is maximal if and only if it has co-dimension one.*

Proof. If $\mathcal{M}$ is a maximal ideal then every element $x \in \mathcal{A}$ can be written as a sum $\lambda e + y$ where $y \in \mathcal{M}$ and $\lambda \in \mathbb{C}$ is unique. This is so, because $\mathcal{A}/\mathcal{M}$ is isometrically isomorphic to $\mathbb{C}$, i.e., any element in $x \in \mathcal{A}$ has the property that there exists a complex number λ such that $\lambda e - x \in \mathcal{M}$. The number λ is unique because if $\mu e - x \in \mathcal{M}$, then $(\lambda - \mu)e = (\lambda e - x) - (\mu e - x) \in \mathcal{M}$ and therfore $\mu = \lambda$. Hence $\mathcal{M}$ has co-dimension one. Conversely, if $\mathcal{M}$ is a closed ideal that has co-dimension one then for any element element $x \in \mathcal{A}$, $x - \lambda e \in \mathcal{M}$ for some unique $\lambda \in \mathbb{C}$. In other words the factor algebra $\mathcal{A}/\mathcal{M}$ is isomorphic to $\mathbb{C}$ and $\mathcal{M}$ is maximal. □

A multiplicative functional is a linear functional $f : \mathcal{A} \to \mathbb{C}$ such that for any $x, y \in \mathcal{A}$ we have $f(xy) = f(x)f(y)$.

Lemma 2.11. *Multiplicative functionals are bounded, in particular if f is any multiplicative function, $|f(x)| \leq \|x\|$, in fact $\|f\| = 1$.*

Proof. We assume that f is not the zero functional. Since $f(x) = f(xe) = f(x)f(e)$ we have that $f(e) = 1$. Further, if x is invertible then $f(e) = f(xx^{-1}) = f(x)f(x^{-1})$ and hence $f(x) \neq 0$. Let x be given and pick $|\lambda| > \|x\|$. Then $(\lambda e - x)$ has an inverse and hence

$$\lambda - f(x) = f(\lambda e - x) \neq 0 \ .$$

This means that for any $|\lambda| > \|x\|$ we have that $f(x) \neq \lambda$. Hence $|f(x)| \leq \|x\|$ and, moreover, $f(e) = 1 = \|e\|$. □

Corollary 2.12. *The kernel of f is a maximal ideal*

Proof. The kernel is closed because f is bounded. It is obviously linear and if $f(x) = 0$ we have that $f(xy) = f(x)f(y) = 0$. Hence the kernel is an ideal. Moreover the co-dimension is one. Hence it is maximal. □

Recall that any linear functional that have the same kernel must be a multiple of each other. To see this suppose that f and g be two bounded linear functionals and assume that $\mathrm{Ker}(f) = \mathrm{Ker}(g)$. Pick any element x_0 such that $f(x_0) \neq 0$ and note that $x - \frac{f(x)}{f(x_0)}x_0$ and note that this element is in the kernel of f. Hence it is in the kernel of g and

$$g(x) = \frac{g(x_0)}{f(x_0)} f(x) \ .$$

Now suppose that the functionals are, in addition, multiplicative. The claim is that $g(x) = f(x)$. Indeed, from the displayed formula we have that

$$1 = g(e) = \frac{g(x_0)}{f(x_0)} f(e) = \frac{g(x_0)}{f(x_0)} \ .$$

Hence $f(x_0) = g(x_0)$ and $f(x) = g(x)$.

Lemma 2.13. *For any maximal ideal $\mathcal{M}$ there exists a unique multiplicative functional such that* $\mathrm{Ker}(f) = \mathcal{M}$. *We denote this functional by $f_{\mathcal{M}}$.*

Proof. The maximal ideal $\mathcal{M}$ has co-dimension one. For any $x \in \mathcal{A}$ there exists a unique $\lambda \in \mathbb{C}$ so that $\lambda e - x \in \mathcal{M}$. Define

$$f(x) = \lambda \ .$$

For $x \in \mathcal{A}$ and $\alpha \in \mathbb{C}$ we have that $\mu e - \alpha x \in \mathcal{M}$ and $\alpha\lambda - \alpha x \in \mathcal{M}$ and hence $\mu = \alpha\lambda$. If $x, y \in \mathcal{A}$ then $(\lambda e - x), (\mu e - x) \in \mathcal{M}$ and hence $(\lambda + \mu) - (x + y) \in \mathcal{M}$. Hence f is linear. If $x, y\mathcal{M}$ then $(\lambda e - x), (\mu e - x) \in \mathcal{M}$ and $(\mu\lambda e - xy) = \mu(\lambda e - x) + x(\mu - y) \in \mathcal{M}$. Hence f is multiplicative. Its kernel is $\mathcal{M}$. Moreover, this functional is uniquely determined by $\mathcal{M}$. □

Thus we have established a one to one correspondence between multiplicative functionals and maximal ideals. The kernel of every multiplicative functional is a maximal ideal and, conversely, if $\mathcal{M}$ is a maximal ideal, it is the kernel of a multiplicative functional.

Definition 2.14. *Denote the set of maximal ideals in a Banach Algebra by $\mathcal{M}(\mathcal{A})$. For every $x \in \mathcal{A}$ we define a map*

$$\widehat{x} : \mathcal{M}(\mathcal{A}) \to \mathbb{C}$$

given by

$$\widehat{x}(\mathcal{M}) = f_{\mathcal{M}}(x)$$

where $f_{\mathcal{M}}$ is the unique multiplicative functional that has $\mathcal{M}$ as its kernel. The map $x \to \widehat{x}$ is called the **Gelfand Transform** *of x.*

Theorem 2.15.
a) The Gelfand Transform is linear, multiplicative and $\widehat{e} = 1$.
b) $x \in \mathcal{M}$ if and only if $\widehat{x}(\mathcal{M}) = 0$. If $\mathcal{M}_1 \neq \mathcal{M}_2$ then there exists $x \in \mathcal{A}$ such that $\widehat{x}(\mathcal{M}_1) \neq \widehat{x}(\mathcal{M}_2)$.
c) $x \in \mathcal{A}$ is invertible if and only if $\widehat{x}(\mathcal{M}) \neq 0$ for every maximal ideal $\mathcal{M}$. In other words $g(x) \neq 0$ for all multiplicative functionals.
d)

$$\sigma(x) = \{\widehat{x}(\mathcal{M}) : \mathcal{M} \in \mathcal{M}(\mathcal{A})\} = \{g(x) : g \text{ is a multiplicative functional}\}$$

e) $|\widehat{x}(\mathcal{M})| \leq 1$ and $\|f_{\mathcal{M}}\| = 1$.

Proof.
a): Let $x, y \in \mathcal{A}$. Then

$$\widehat{xy}(\mathcal{M}) = f_{\mathcal{M}}(xy) = f_{\mathcal{M}}(x) f_{\mathcal{M}}(y) = \widehat{x}(\mathcal{M})\widehat{y}(\mathcal{M}) \ .$$

The linearity is obvious, as well as $\widehat{e} = 1$.
b): If $x \in \mathcal{M}$, then $f_{\mathcal{M}}(x) = 0$ and conversely. Hence $\widehat{x} = 0$ if and only if $x \in \mathcal{M}$. If $\mathcal{M}_1 \neq \mathcal{M}_2$ then $f_{\mathcal{M}_1} \neq f_{\mathcal{M}_2}$ and conversely which proves the statement.
c): If x is invertible then x is not a member of any maximal ideal and conversely. Because every multiplicative functional g has a kernel that is a maximal ideal and hence $g(x) \neq 0$. Conversely if $g(x) \neq 0$, x cannot be in any maximal ideal.
d): The two sets in item d) are the same because there is a one to one correspondence between multiplicative functionals and maximal ideals. Denote this set by $\mathcal{S}$. Let $x \in \mathcal{A}$ and $\lambda \in \sigma(x)$. Then there exists a multiplicative functional g with $g(\lambda e - x) = 0$ and hence $\lambda = g(x)$ and hence $\lambda \in \mathcal{S}$. Conversely assume if f is a multiplicative functional and set $f(x) =: \lambda$ and note that $f(\lambda e - x) = 0$ and hence $\lambda e - x$ is not invertible and $\lambda \in \sigma(x)$.
e): This statement follows from a lemma we proved before. ☐

Next we have to talk a bit about the topology of the set of all multiplicative functionals. These are elements in $\mathcal{A}^*$ and there is a natural topology on $\mathcal{A}^*$, w^*-topology (weak star). Let us recall these concepts in the case of separable spaces which allows us to talk in terms of sequences. Otherwise we have to talk about nest or directly in terms of neigborhoods. On a Banach space we usually consider two different topologies, the strong topology in which sequences converge strongly and the weak topology in which a sequence $x_n \in \mathcal{A}$ converges to x weakly if and only if $f(x_n) \to f(x)$ for all $f \in \mathcal{A}^*$. The dual space $\mathcal{A}^*$ is also a Banach Space and hence we have the strong convergence and the weak convergence. Once more, $f_n \to f$ weakly if and only if for every $g \in \mathcal{A}^{**}$ we have that $g(f_n) \to g(f)$. There is, however a weaker topology. There is a class of bounded linear functional given by $f \to f(x)$. Hence we say that f_n converges w^* to f if and only if $f_n(x) \to f(x)$ for all $x \in \mathcal{A}$. The standard definition is

Definition 2.16. *The $\mathcal{Z}$ be a family of linear functionals that separate points in $\mathcal{A}$. The the Z-weak topology $\sigma(\mathcal{A}, Z)$ is the weakest topology in which all linear functionals in Z are continuous.*

Because of the separation property this topology is a Hausdorff topology i.e., for any $f, g \in Z$ with $f \neq g$ there exist open sets O_f, O_g with $O_f \cap O_g = \emptyset$ and $f \in O_f, g \in O_g$. Thus, $\sigma(\mathcal{A}, \mathcal{A}^*)$ is the weak topology and $\sigma(\mathcal{A}^*, \mathcal{A})$ is the w^*-topology on $\mathcal{A}^*$. The system of neighborhoods that generate that topology is given by

$$O_{x_1,\ldots,x_n,\delta}(f_0) = \{f \in \mathcal{A}^* : |f(x_k) - f_0(x_k)| < \delta, k = 1, \ldots, n\} .$$

The key theorm about the w^*-topology is the following

Theorem 2.17. *Let $\mathcal{A}^*$ be the dual of some Banach space $\mathcal{A}$. Then the unit ball in $\mathcal{A}^*$ is w^* compact.*

Note that in the case of a reflexive Banach space the weak topology and the weak star topology are the same.

Lemma 2.18. *The space of multiplicative functionals is a closed subset of $\mathcal{A}^*$. The function $f_{\mathcal{M}} \to f_{\mathcal{M}}(x)$ is a continuous function on $\mathcal{A}^*$, i.e., $\widehat{x}(\mathcal{M})$ is a continuous function on $\mathcal{M}(\mathcal{A})$.*

Proof. Let f_0 be in the closure of the multiplicative functionals. Thus, in every neighborhood of f_0 we find a multiplicative functional. Recall that every multiplicative functional is of the form $f_{\mathcal{M}}$ for some maximal ideal $\mathcal{M}$. Now we pick the neighborhood $O_{x,y,x+y,\delta}(f_0)$ and note that this means

$$|f_{\mathcal{M}}(x) - f_0(x)| < \delta \ , |f_{\mathcal{M}}(y) - f_0(y)| < \delta \ , |f_{\mathcal{M}}(x+y) - f_0(x+y)| < \delta \ .$$

Hence $|f_0(x+y)-f_0(x)-f_0(y)| = |f_0(x+y)-f_{\mathcal{M}}(x+y)-(f_0(x)-f_{\mathcal{M}}(x))-(f_0(y)-f_{\mathcal{M}}(y)| \leq 3\delta$. and since $\delta > 0$ is arbitrary, $f_0(x+y) = f_0(x) + f_0(y)$. A similar argument shows that $f_0(\alpha x) = \alpha f_0(x)$ and that $f_0(x)f_0(y) = f_0(xy)$. Hence f_0 is a multiplicative functional and hence this set is w^* closed. For a given multiplicative functional $f_{\mathcal{M}_0}$ we pick the neighborhood $O_{x_0,\varepsilon}(f_{\mathcal{M}_0})$. If $f_{\mathcal{M}} \in O_{x_0,\varepsilon}(f_{\mathcal{M}_0})$ then

$$|\widehat{x}_0(\mathcal{M}_0) - x_0(\mathcal{M})| = |f_{\mathcal{M}_0}(x_0) - f_{\mathcal{M}}(x_0)| < \varepsilon$$

and hence $\widehat{x}_0$ is continuous at $\mathcal{M}_0$. □

we may collect what we have so far in the following theorem:

Theorem 2.19. *The map $x \to \widehat{x}$ is a homomorphism of the Banach Algebra $\mathcal{A}$ into the algebra $C(\mathcal{M}(\mathcal{A}))$, the continuous functions on the compact Hausdorff space $\mathcal{M}(\mathcal{A})$ the space of maximal ideals.*

We may sharpen this theorem in a few ways by introducing the following notions.

Definition 2.20. *The intersection of all maximal ideals is called the* **radical** *of $\mathcal{A}$. A Banach Algebra is called* **regular** *if for any $x \in \mathcal{A}$, $\|x^2\| = \|x\|^2$. A Banach Algebra $\mathcal{A}$ is* **symmetric** *if for any $\widehat{x}(\mathcal{M})$ there exists $y \in \mathcal{A}$ such that $\widehat{y}(\mathcal{M}) = \overline{\widehat{x}(\mathcal{M})}$.*

Theorem 2.21.
a) If the radical of a Banach Algebra consists only of the the zero element, then the map $x \to \widehat{x}$ is one to one.
b) Is $\mathcal{A}$ regular, then is is isometrical isomorphic to $C(\mathcal{M}(\mathcal{A}))$.

c) Is $\mathcal{A}$ symmetric, the the image of $\mathcal{A}$ under the map $x \to \widehat{x}$ is dense in $C(\mathcal{M}(\mathcal{A}))$.
d) If $\mathcal{A}$ is regular and symmetric, then it is isometrically isomorphic to $C(\mathcal{M}(\mathcal{A}))$.

Proof.
a): Assume that there exists $x_0 \neq 0$ with $f(x_0) = 0$ for all multiplicative functionals. Then $x_0 \in \mathcal{M}$ for all maximal ideals and hence it is in the radical and thus zero. This is a contradiction.
b): By the assumed regularity we have that

$$\|x^{2^n}\| = \|x\|^{2^n}$$

and hence

$$\|x^{2^n}\|^{1/2^n} = \|x\|$$

and so $\sigma(x) = \|x\|$. Thus the radical consists only of the zero element. Hence the map $x \to \widehat{x}$ is an isomorphism. We also remember that

$$\max_{\mathcal{M}} |\widehat{x}(\mathcal{M})| = \sigma(x) = \|x\|$$

which yields the isometry.
c): Let A consist of the functions $\widehat{x}(\mathcal{M})$. Since $e(\mathcal{M}) = 1$, for $\mathcal{M}_1 \neq \mathcal{M}_2$ we have x such that $\widehat{x}(\mathcal{M}_1) \neq \widehat{x}(\mathcal{M}_2)$ and since $\mathcal{A}$ is symmetric we have that A is dense in $C(\mathcal{M}(\mathcal{A}))$. Thus A is dense in $C(M)$ by the Stone-Weierstrass Theorem.
d): The map $x \to \widehat{x}$ is an isometric isomorphism. It is symmetric and hence the set $\{\widehat{x}(\mathcal{M}) : x \in \mathcal{A}\}$ is dense in $C(\mathcal{M}(\mathcal{A}))$. Since $\mathcal{A}$ is complete, so is $\{\widehat{x}(\mathcal{M}) : x \in \mathcal{A}\}$ and hence this set is equal to $C(\mathcal{M}(\mathcal{A}))$.

$\square$

Theorem 2.22 (Stone-Weierstrass theorem)**.** *Consider the Banach Algebra of continuous functions on a compact set M. Let A be a sub-algebra such that $1 \in A$, A separates points, i.e., for $t_1 \neq t_2 \in M$ there exists $a \in A$ such that $a(t_1) \neq a(t_2)$ and finally assume that if $a(t) \in C(M)$ then $\overline{a(t)} \in C(M)$. Then A is dense in $C(M)$.*