NAME:

PRACTICE TEST 1 FOR MATH 2551 F1-F4, SEPTEMBER 20, 2018

This test should be taken without any notes and calculators. Time: 50 minutes. Show your work, otherwise credit cannot be given.

Problem 1: Compute the volume of the parallelepiped spanned by the vectors

$$\langle 1,2,2\rangle \ , \langle 2,1,-2\rangle \ , \ \langle 1,1,1\rangle$$

$$\langle 1,2,2\rangle \times \langle 2,1,-2\rangle = 3\langle -2,2,-1\rangle$$

so that the volume is

$$|3\langle -2,2,-1\rangle \cdot \langle 1,2,1\rangle| = 3 \ .$$

Problem 2: Find the distance between the point $(1,2,3)$ and the plane

$$x + 2y + 3z = 6 \ .$$

Pick any point on the plane, e.g., $(6,0,0)$, then form the vector $(6,0,0) - (1,2,3) = \langle 5,-2,-3\rangle$. Then project this vector onto the normal vector to the plane given by $\langle 1,2,3\rangle$. The result is $-\frac{4}{7}\langle 1,2,3\rangle$ whose length is $\frac{4\sqrt{14}}{7}$.

Problem 3: Given the curve

$$\vec{r}(t) = \langle e^t, t, t^2\rangle \ , t \in \mathbb{R} \ .$$

Find the line tangent to the curve at the point $\langle e,1,1\rangle$, i.e., at $t=1$.

The derivative is $\vec{r}'(t) = \langle e^t, 1, 2t\rangle$ so that $\vec{r}'(1) = \langle e,1,2\rangle$. Thus the direction of the tangent line is $\langle e,1,2\rangle$. Since the line has to pass through the point $\vec{r}(1) = (e,1,1)$ we get for this line

$$x = e + es \ , \ y = 1 + s \ , \ z = 1 + 2s$$

where s is the parameter describing the line.

Problem 4: A particle has the trajectory

$$\vec{r}(t) = \langle t^2/2, t, e^t\rangle$$

Find the tangential acceleration a_T, the normal acceleration a_N as well as $\vec{T}$, $\vec{N}$ and $\vec{B}$.

The velocity is $\vec{v}(t) = \langle t, 1, e^t\rangle$ so that the speed is $s(t) = \sqrt{t^2+1+e^{2t}}$. We have that

$$a_T = s'(t) = \frac{t + e^{2t}}{\sqrt{t^2+1+e^{2t}}}$$

Further $\vec{a} = \langle 1, 0, e^t\rangle$ so that $|\vec{a}|^2 = 1 + e^{2t}$ so that

$$a_N^2 = 1 + e^{2t} - \frac{(t+e^{2t})^2}{t^2+1+e^{2t}}$$

or

$$a_N = \sqrt{1 + e^{2t} - \frac{(t+e^{2t})^2}{t^2+1+e^{2t}}}$$

$$\vec{T} = \frac{\langle t, 1, e^t \rangle}{\sqrt{t^2+1+e^{2t}}}$$

$$\vec{T}' = \frac{\langle 1, 0, e^t \rangle}{\sqrt{t^2+1+e^{2t}}} - \frac{\langle t, 1, e^t \rangle (t+e^{2t})}{(t^2+1+e^{2t})^{3/2}} = \frac{\langle 1+e^{2t}(1-t), -(t+e^{2t}), e^t(1-t+t^2) \rangle}{(t^2+1+e^{2t})^{3/2}}$$

so that

$$\vec{N} = \frac{\langle 1+e^{2t}(1-t), -(t+e^{2t}), e^t(1-t+t^2) \rangle}{\sqrt{(1+e^{2t}(1-t))^2 + (t+e^{2t})^2 + e^{2t}(1-t+t^2)^2}}$$

and

$$\vec{B} = \vec{T} \times \vec{N} = \frac{\langle e^t, e^t(1-t), -1 \rangle}{\sqrt{e^{2t}(1+(1-t)^2)+1}}$$

Problem 5: Calculate the arc length of the curve $\vec{x}(t) = \langle t^2, t^3 \rangle$ where t ranges from 0 to 1.

We compute $\vec{r}'(t) = \langle 2t, 3t^2 \rangle$ so that $|\vec{r}'(t)| = \sqrt{4t^2+9t^4} = t\sqrt{4+9t^2}$. The length is given by

$$L = \int_0^1 2t\sqrt{1+\frac{9t^2}{4}}dt$$

which by substituting $u = t^2$ is the integral

$$= \int_0^1 \sqrt{1+\frac{9u}{4}}du = \frac{4}{9}\frac{2}{3}(1+\frac{9u}{4})^{3/2}\Big|_0^1 = \frac{8}{27}[(1+\frac{9}{4})^{3/2} - 1]$$

Problem 6: A real valued function $f(\vec{x})$ on some domain $D \in \mathbb{R}^2$ satisfies the inequality

$$|f(\vec{x}) - f(\vec{x}_0)| \le 2\sqrt{|\vec{x} - \vec{x}_0|}$$

for all $\vec{x} \in D$ where $\vec{x}_0$ is some fixed point in D. For any given $\varepsilon > 0$ find $\delta > 0$ so that

$$|f(\vec{x}) - f(\vec{x}_0)| < \varepsilon$$

whenever $|\vec{x} - \vec{x}_0| < \delta$.

We know that

$$|f(\vec{x}) - f(\vec{x}_0)| \le 2\sqrt{|\vec{x} - \vec{x}_0|}$$

and hence if $2\sqrt{|\vec{x} - \vec{x}_0|} < \varepsilon$ we have that

$$|f(\vec{x}) - f(\vec{x}_0)| < \varepsilon \ .$$

Thus if we choose δ such that $2\sqrt{\delta} = \varepsilon$, i.e.,

$$\delta = (\frac{\varepsilon}{2})^2$$

we have that whenever $|\vec{x} - \vec{x}_0| < \delta$ then $|f(\vec{x}) - f(\vec{x}_0)| < \varepsilon$.

Problem 7: Consider the function

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}} \ .$$

For $(x,y,z) \neq (0,0,0)$ compute f_x, f_y, f_z and $f_{xx}+f_{yy}+f_{zz}$.

We have that

$$f_x = -\frac{x}{(x^2+y^2+z^2)^{3/2}} \ , \ f_y = -\frac{y}{(x^2+y^2+z^2)^{3/2}} \ , \ f_z = -\frac{z}{(x^2+y^2+z^2)^{3/2}}$$

Further

$$f_{xx} = -\frac{1}{(x^2+y^2+z^2)^{3/2}} + 3\frac{x^2}{(x^2+y^2+z^2)^{5/2}} = \frac{3x^2-x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}} \ .$$

In the same fashion we get that

$$f_{yy} = \frac{3y^2-x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}} \ , \ f_{zz} = \frac{3z^2-x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}} \ .$$

If we sum these up we get that

$$f_{xx}+f_{yy}+f_{zz} = 0 \ .$$

Problem 8: Sketch the level curve of of the function $\sqrt{x+y^2-3}$ that passes through the point $(3,-1)$.

At the point $(3-1)$ the value of the function is $\sqrt{3+1-3} = 1$. Solving $\sqrt{x+y^2-3} = 1$ we get that $x+y^2-3 = 1$ or $x+y^2 = 4$. Now sketch the parabola $x = 4-y^2$.