NAME:

PRACTICE TEST 2 FOR MATH 2551 F1-F4, OCTOBER 31, 2018

This test should be taken without any notes and calculators. Time: 50 minutes. Show your work, otherwise credit cannot be given.

Problem 1: Sketch the level curve at height $c = 1$ for the function

$$f(x, y, z) = z(x^2 + y^2)^{-1/2} \ .$$

Solution: The equation

$$z(x^2 + y^2)^{-1/2} = 1$$

rewritten as $z = (x^2 + y^2)^{1/2}$ shows that the level surface is a cone in the upper half space with apex at the origin generated by rotating the line $z = y$ around the z-axis.

Problem 2: Find the unit vector in the direction in which f increases most rapidly at P

$$f(x, y) = y^2 e^{2x} \ , \quad P : (0, 1) \ .$$

Solution: The direction of most rapid increase is the direction of the gradient.

$$\nabla f(x, y) = \langle 2y^2 e^{2x}, 2ye^{2x} \rangle \ ,$$

so that $\nabla f(0, 1) = \langle 2, 2 \rangle$ and hence the unit vector is given by

$$\frac{1}{\sqrt{2}} \langle 1, 1 \rangle \ .$$

Problem 3: Find an equation for the plane tangent to the graph of the function $f(x, y) = (x^2 + y^2)^2$ at the point $(1, 1, 4)$.

Solution: Quite generally the equation for a plane tangent at the point $(x_0, y_0, f(x_0, y_0))$ is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \ .$$

We have $f_x = 4x(x^2 + y^2), f_y = 4y(x^2 + y^2), f(1, 1) = 4$ so that the equation is given by

$$z = 4 + 8(x - 1) + 8(y - 1) \ .$$

Problem 4: Find the absolute extreme values taken by the function f on the domain R

$$f(x, y) = (x - 3)^2 + y^2 \ , \ R : 0 \leq x \leq 4 \ , x^2 \leq y \leq 4x \ .$$

Solution: First we have to look for the critical points of f in the domain R. $f_x = 2(x-3) = 0, f_y = 2y = 0$ and hence $(3,0)$ is a candidate for a critical point but it is outside the region R. Hence the maximum nor the minimum is attained inside the region R. Next we have to check the function on the boundary of R. On the line $y = 4x$ the function takes the values $g(x) = f(x, 4x) = (x-3)^2 + 16x^2 = 17x^2 - 6x + 9$, $0 \le x \le 4$. We have $g'(x) = 34x - 6 = 0$ and hence the point $(3/17, 12/17)$ is a candidate. Next we look at the curve $y = x^2$ and we have to analyze the function $h(x) = f(x, x^2) = (x-3)^2 + x^4$ on the interval $(0,4)$. Again, $h'(x) = 4x^3 + 2x - 6 = 0$ and we see right away that $x = 1$ is a root and by long division we find that $4x^3 + 2x - 6 = (x-1)(4x^2 + 2x + 6)$ and hence $x = 1$ is the only root in $(0,4)$. Thus we have a second candidate $(1,1)$. To this list we have to add the corners: $(4,16)$ and $(0,0)$. We have the following values:

$$f(0,0) = 9 \ , \ f(4,16) = 257 \ , \ f(\frac{3}{17}, \frac{12}{17}) = \frac{144}{17} \ , \ f(1,1) = 5 \ .$$

Clearly the maximum value is 257 attained at the point $(4,16)$ and the minimum value is 5 attained at the point $(1,1)$.

Problem 5: Find the points on the sphere $x^2 + y^2 + z^2 = 1$ that are closest and farthest away from the point $(2,1,2)$.

Solution: We have to use Lagrange multipliers. We set $f(x,y,z) = (x-2)^2 + (y-1)^2 + (z-2)^2$ which is the square of the distance to be minimized. The constraint is $g(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$. The equations $\nabla f = \lambda \nabla g$ yield

$$(x-2) = \lambda x \ , \ (y-1) = \lambda y \ , \ (z-2) = \lambda z$$

and $\lambda \neq 1$. Hence we find

$$x = \frac{2}{1-\lambda} \ , \ y = \frac{1}{1-\lambda} \ , \ z = \frac{2}{1-\lambda}$$

and λ has to be chosen such that this point satisfies the constraint which yields

$$(1-\lambda)^2 = 9.$$

There are two solutions $\lambda_1 = 4$ and $\lambda_2 = -2$. This yields the two points

$$-\frac{1}{3}(2,1,2) \ , \ \frac{1}{3}(2,1,2) \ .$$

The first is farthest away and the second is the closest to the point $(2,1,2)$. A little bit of geometry confirms this result.

Problem 6: Find the volume of the intersection of the ball of radius R centered at the origin and the cylinder $(x - \frac{R}{2})^2 + y^2 = \frac{R^2}{4}$.

Solution: The ball of radius R is bounded by the sphere $x^2 + y^2 + z^2 = R^2$. The equation for the cylinder can be written as

$$x^2 + y^2 - xR = 0 \ .$$

We have to compute

$$\int\int_D\int dV$$

where D is the domain inside the sphere $x^2+y^2+z^2 \le R^2$ intersected with the interior of the cylinder $x^2+y^2-xR \le 0$. Let's try first in terms of Cartesian coordinates:

$$\int_0^R \int_{-\sqrt{xR-x^2}}^{\sqrt{xR-x^2}} \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} dzdydx \ .$$

This iterated integral is rather complicated and hence we try using polar coordinates. Set $x = r\cos\theta$ and $y = r\sin\theta$. The condition to be inside the sphere means that $-\sqrt{R^2-r^2} \le z \le \sqrt{R^2-r^2}$. and the condition for being inside the cylinder is written as

$$r^2 - r\cos\theta R \le 0 \ ,$$

or

$$r \le R\cos\theta \ .$$

We see that r ranges from 0 to $R\cos\theta$ and θ ranges from $-\pi/2$ to $\pi/2$. The integral is then

$$\int_{-\pi/2}^{\pi/2}\int_0^{R\cos\theta} 2\sqrt{R^2-r^2}rdrd\theta \ .$$

The substitution $u = r^2$ yields the integral

$$\int_0^{R^2\cos^2\theta} \sqrt{R^2-u}du = -\frac{2}{3}(R^2-u)^{3/2}\Big|_0^{R^2\cos^2\theta} = \frac{2R^3}{3}[1-(1-\cos\theta^2)^{3/2}] = \frac{2R^3}{3}[1-|\sin\theta|^3]$$

Notice we have to put $|\sin\theta|$! It remains to integrate

$$\frac{2R^3}{3}\int_{-\pi/2}^{\pi/2}[1-|\sin\theta|^3]d\theta$$

The integral

$$\int_{-\pi/2}^{\pi/2}|\sin\theta|^3 d\theta = 2\int_0^{\pi/2}\sin^3\theta d\theta = 2\int_0^{\pi/2}(1-\cos^2\theta)\sin\theta d\theta$$

and the substitution $u = \cos\theta$ yields

$$2\int_0^1(1-u^2)du = \frac{4}{3}$$

Thus, the volume is

$$\frac{8R^3}{9} \ .$$

Problem 7: Find the area of the region bounded by $x = y^{1/2}$ and by $x = y^4$.

Solution: The intersection of the two curves is given by the points $(1,1)$ and $(0,0)$. On the interval $0 \le y \le 1$ the curve $x = y^{1/2}$ is above the curve $x = y^4$ and we have for the area the integral

$$\int_0^1 \int_{y^4}^{y^{1/2}} dxdy = \int_0^1 [y^{1/2} - y^4]dy = \frac{2}{3}y^{3/2} - \frac{1}{5}y^5]\Big|_0^1 = \frac{7}{15}$$

Problem 8: Compute the integral $\int\int_R (x^4 - 2y)dA$ where $R = \{(x, y) : -1 \leq x \leq 1\ ,\ -x^2 \leq y \leq x^2\}$.

Solution: We compute

$$\int_{-1}^1 \int_{-x^2}^{x^2} (x^4 - 2y)dydx = \int_{-1}^1 [x^4 y - y^2]\Big|_{-x^2}^{x^2} dx = 2\int_{-1}^1 x^6 dx = \frac{4}{7}$$