# MATRICES ARE SIMILAR TO TRIANGULAR MATRICES

## 1. Complex matrices

Recall that the complex numbers are given by $a + ib$ where $a$ and $b$ are real and $i$ is the imaginary unity, i.e.,

$$i^2 = -1 \ .$$

In what we describe below, $A$ is an $n \times n$ matrix, not necessarily real. The set of complex numbers is denoted by $\mathbb{C}$. The addition and multiplication of two complex numbers are given by

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$
$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

and the usual rules apply. In particular complex multiplication is commutative, i.e., $z_1 z_2 = z_2 z_1$.

Notable is the inverse, i.e., the complex number $z$ that solves the equation $(a + ib)z = 1$. The solution is given by

$$z = \frac{(a - ib)}{a^2 + b^2} \ .$$

If a complex number $z = a + ib$, then the **complex conjugate** $\overline{z} = a - ib$. It has the property that

$$\overline{z}z = |z|^2 = a^2 + b^2 \ .$$

If an $n \times m$ matrix $A$ has complex entries, we call it a complex matrix. Thus if $A$ is a given complex matrix and $\vec{b}$ is a given complex vector we may try to solve the equation

$$A\vec{x} = \vec{b} \ .$$

Clearly $\vec{x}$ will be a complex vector. The row reduction algorithm works also in this situation and we can talk about the null space of a matrix the column space. These are now complex vector spaces. One has to think a little about the other two subspaces and this has to do with what we mean by a dot product for complex vectors. Now one is tempted to define the dot product between two complex vectors $\vec{z}$ and $\vec{w}$ in $\mathbb{C}^n$ by $z_1 w_1 + z_2 w_2 + \cdots z_n w_n$. Here is the problem: consider the vector in $\mathbb{C}^2$

$$\vec{z} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

then

$$\vec{z} \cdot \vec{z} = 1^2 + i^2 = 1 - 1 = 0 \ .$$

Recall that for real vectors we interpreted the dot of a vector with itself as the square of the length. In the complex domain this does not seem to work. If we use the complex conjugate, however, the story looks more promising. We define the **inner product** of two vectors $\vec{w}, \vec{z} \in \mathbb{C}^n$ by

$$\langle \vec{w}, \vec{z} \rangle = w_1 \overline{z}_1 + \cdots w_n \overline{z}_n \ .$$

Note that we have that

$$\langle \vec{z}, \vec{z} \rangle = |z_1|^2 + \cdots + |z_n|^2$$

which is strictly positive unless $\vec{z}$ is the zero vector. Hence we may define

$$\|\vec{z}\| = \sqrt{\langle \vec{z}, \vec{z} \rangle} \ .$$

The name 'inner product' is used to distinguish it from the dot product. Note that for real vectors the inner product reduces to the dot product. There is one thing one has to be careful about. We have

$$\langle \vec{z}, \vec{w} \rangle = \overline{\langle \vec{w}, \vec{z} \rangle} \ ,$$

which is different from the corresponding relation for the dot product. There are a number of simple consequences. One is that Schwarz's inequality still holds,

$$|\langle \vec{z}, \vec{w} \rangle| \le \|\vec{z}\| \|\vec{w}\| \ .$$

Further the triangle inequality is also true

$$\|\vec{z} + \vec{w}\| \le \|\vec{z}\| + \|\vec{w}\| \ .$$

These things are easy to proof. Do these as an exercise.

Now we define two vectors $\vec{z}$ and $\vec{w}$ to be **orthogonal** if

$$\langle \vec{z}, \vec{w} \rangle = 0 \ .$$

Note that in this definition it is irrelevant whether we consider $\langle \vec{z}, \vec{w} \rangle = 0$ or $\langle \vec{w}, \vec{z} \rangle = 0$, it amounts to the same.

We define the **adjoint** of a matrix $A$ the matrix $\overline{A^T}$ and denote it by $A^*$. Thus the adjoint is found by taking the the complex conjugate of all the matrix elements and then take the transpose or, what amounts to the same, the transpose and then taking the complex conjugate of all its matrix elements. As before one easily finds that $(AB)^* = B^* A^*$ and that $(A^*)^{-1} = (A^{-1})^*$. It is useful to observe that for any complex vectors $\vec{z}$ and $\vec{w}$ we have that

$$\langle \vec{z}, A\vec{w} \rangle = \langle A^* \vec{z}, \vec{w} \rangle \ . \tag{1}$$

We have

**Theorem 1.1.** *Let $A$ be an $n \times m$ matrix. The the null space $N(A)$ is a subspace of $\mathbb{C}^m$, $C(A)$ a subspace of $\mathbb{C}^n$. Similarly $N(A^*)$ is a subspace of $\mathbb{C}^n$ and $C(A^*)$ a subspace of $\mathbb{C}^m$. Moreover,*

$$C(A) \oplus N(A^*) = \mathbb{C}^n$$

*and*

$$C(A^*) \oplus N(A) = \mathbb{C}^m$$

*where $\oplus$ denotes the orthogonal sum. In other words the orthogonal complement of $C(A)$ in $\mathbb{C}^n$ is $N(A^*)$ and the orthogonal complement of $C(A^*)$ in $\mathbb{C}^m$ is $N(A)$.*

*Proof.* The orthogonal complement of $C(A)$ consists of vectors in $\mathbb{C}^n$ that are orthogonal to all the column vectors of the matrix $A$ which is the same as all the vectors that are perpendicular to each row of the matrix $A^*$. Hence $C(A)^\perp = N(A^*)$. That $C(A^*)^\perp = N(A)$ follows in the same fashion. □

The next problem is whether there is a notion of least squares approximation for complex matrices. As before we would like to find $\vec{x} \in \mathbb{C}^m$ such that $A\vec{x} - \vec{b}$ has the smallest length. In other words we want to minimize

$$\|A\vec{x} - \vec{b}\|^2 \ .$$

We claim that the optimizing $\vec{x}$ is such that $A\vec{x} - \vec{b}$ is perpendicular to the column space of $A$. Assuming this, we have that $A\vec{x} - \vec{b}$ must be in the null space of $A^*$, i.e.,

$$A^* A\vec{x} = A^*\vec{b} \ .$$

This are once again the normal equations. What about projections? Assume that $V$ is a complex subspace of $\mathbb{C}^n$ whose complex dimension is $m$. We choose a basis of in general complex vectors in $V$ and create a matrix with these vectors as column vectors. Given a vector $\vec{b} \in \mathbb{C}^n$ we can try to write this vector as a vector in $V$, i.e., the column space of $A$ and a vector perpendicular to $V$. I.e., we have to find $\vec{x}$ so that

$$A\vec{x} - \vec{b} \perp V \ .$$

As before $A\vec{x} - \vec{b} \in N(A^*)$ and hence

$$A^* A\vec{x} = A^*\vec{b} \ .$$

The matrix $A^*A$ is $m \times m$ and has rank $m$ and hence is invertible. Hence

$$\vec{x} = (A^*A)^{-1}A^*\vec{b}$$

and the projection of $\vec{b}$ onto $V$ is given by

$$A\vec{x} = A(A^*A)A^*\vec{b}$$

The matrix $P = A(A^*A)A^*$ is easily seen to be a projection. In fact we also have that $P^* = P$.

## 2. Unitary Matrices

The unitary matrices are the analog of complex matrices but in $\mathbb{C}^n$. Imagine you are given $n$ orthonormal vectors $\vec{u}_1, \dots, \vec{u}_n$

$$\langle \vec{u}_k, \vec{u}_\ell \rangle = 0 \ , k \neq \ell$$

$$\langle \vec{u}_k, \vec{u}_k \rangle = 1 \ , k = 1, \dots, n \ .$$

The we can form

$$U = \begin{bmatrix} \vec{u}_1 & \dots & \vec{u}_n \end{bmatrix}$$

It is now easy to check that

$$U^*U = I_n$$

and hence

$$U^*U = I_n = UU^* \ ,$$

because the matrix $U$ has full rank and hence is invertible. Such matrices are called unitary matrices. It is straightforward to imitate the Gram-Schmidt procedure and hence any complex matrix can be written as

$$A = UR$$

where $U$ has column vectors that are orthonormal (maybe not $n$ of them) and $R$ is an upper triangular matrix. Of course, $R$ is complex. As a consequence, the column vectors of $U$ form an orthonormal basis for the column space of $A$. The projection onto this column space is given by $UU^*$ (note that $U^*U = I$.). The least square solution is then given by

$$R\vec{x} = Q^*\vec{b} \ .$$

## 3. Eigenvalues and Eigenvectors

Recall that a nonzero vector $\vec{v}$ is an eigenvector of the $n \times n$ matrix $A$, if

$$A\vec{v} = \lambda\vec{v} \ .$$

The number $\lambda$ is called the eigenvalue. We know that if $\vec{v}$ is an eigenvector then $A - \lambda I$ is a singular matrix and hence

$$\det(A - \lambda I) = 0 \ .$$

The characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$ is a polynomial with complex coefficients and hence we can factor it into linear factors

$$p_A(\lambda) = (-1)^n(\lambda - \lambda_1)\cdots(\lambda - \lambda_n)$$

where the roots are the eigenvalues. The following theorem is a precursor to what is known as Schur factorization, but somewhat simpler.

**Theorem 3.1.** *Let $A$ be any complex $n \times n$ matrix. There exists an invertible $n \times n$ matrix $V$ and an upper triangular matrix $T$ such that*

$$A = VTV^{-1} \ .$$

*The diagonal elements of $T$ are precisely the eigenvalues.*

*Proof.* The matrix $A$ has an eigenvalue $\lambda_1$ and hence there exists a vector $\vec{v}_1 \neq \vec{0}$ such that $A\vec{v}_1 = \lambda_1\vec{v}_1$. Now extend $\vec{v}_1$ to a basis by choosing vectors $\vec{w}_2, \dots, \vec{w}_n$ such that $\vec{v}_1, \vec{w}_2, \dots, \vec{w}_n$ form a basis for $\mathbb{C}^n$. Now Consider the matrix $V_1 = [\vec{v}_1, \vec{w}_2, \dots, \vec{w}_n]$ and note that

$$AV_1 = [\lambda_1\vec{v}_1, A\vec{w}_2, \dots, A\vec{w}_n] \ .$$

We are going to write the right side as

$$V_1V_1^{-1}\left[\lambda_1\vec{v}_1, A\vec{w}_2, \dots, A\vec{w}_n\right] = V_1\left[\lambda_1V^{-1}\vec{v}_1, A\vec{w}_2, \dots, A\vec{w}_n\right] \ .$$

The matrix $V_1$ is invertible (why?). The vector $\lambda_1V^{-1}\vec{v}_1$ can be written as

$$\begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{bmatrix}$$

and hence the first column of the vector

$$V_1\begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{bmatrix} = c_1\vec{v}_1 + c_2\vec{w}_2 + \cdots + c_n\vec{w}_n$$

must be equal to $\lambda_1\vec{v}_1$. Since the vectors $\vec{v}_1, \vec{w}_2, \dots, \vec{w}_n$ are linearly independent we must have that $c_1 = \lambda_1$ and all the other c's equal to 0. Hence

$$\left[\lambda_1V^{-1}\vec{v}_1, A\vec{w}_2, \dots, A\vec{w}_n\right] = [\lambda_1\vec{e}_1, A\vec{w}_2, \dots, A\vec{w}_n]$$

where $\vec{e}_1$ is the first element of the canonical basis. Thus, we have shown that

$$AV_1 = V_1 T_1$$

where $T_1$ is of the form

$$\begin{bmatrix} \lambda_1 & * & \dots & * \\ 0 & * & \dots & * \\ 0 & * & \dots & * \\ 0 & * & \dots & * \end{bmatrix}$$

We write

$$T_1 = \begin{bmatrix} \lambda_1 & \vec{*}^T \\ \vec{0} & A_1 \end{bmatrix}$$

where $A_1$ is an $(n-1) \times (n-1)$ matrix and $\vec{*}$ denotes some vector whose precise form is irrelevant for our purposes. Now we apply the same procedure again to the matrix $A_1$ and we find an invertible $(n-1) \times (n-1)$ matrix $W_1$ such that

$$A_1 W_2 = W_2 S_2$$

where

$$S_2 = \begin{bmatrix} \mu_2 & \vec{*}^T \\ 0 & A_2 \end{bmatrix} ,$$

where $A_2$ is an $(n-2) \times (n-2)$ matrix and $\mu_2$ is an eigenvalue of $A_1$. Define

$$V_2 = \begin{bmatrix} 1 & * \\ 0 & W_2 \end{bmatrix}$$

and note that

$$T_1 V_2 = \begin{bmatrix} \lambda_1 & \vec{*}^T \\ \vec{0} & A_1 W_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \vec{*}^T \\ \vec{0} & W_2 S_2 \end{bmatrix} = V_2 \begin{bmatrix} \lambda_1 & \vec{*}^T \\ \vec{0} & S_2 \end{bmatrix}$$

Thus,

$$AV_1 V_2 = V_1 T_1 V_2 = V_1 V_2 T_2$$

where

$$T_2 = \begin{bmatrix} \lambda_1 & \vec{*}^T \\ \vec{0} & S_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & * & \vec{*}^T \\ 0 & \mu_2 & \vec{*}^T \\ \vec{0} & \vec{0} & A_2 \end{bmatrix} .$$

Continuing in this fashion we arrive at

$$AV_1 \cdots V_n = V_1 \cdots V_n T$$

where $T$ is upper triangular. The matrix $V = V_1 \cdots V_n$ is invertible. The fact that the diagonal elements of $T$ are the eigenvalues follows from the relation

$$\det(A - \lambda I) = \det(VTV^{-1} - \lambda I) = \det V(T - \lambda I)V^{-1} = \det(T - \lambda I)$$

and the fact that the determinant of an upper triangular matrix is the product of the diagonal elements. □

Here are two consequences.

**Theorem 3.2.** *Let $A$ be an $n \times n$ matrix and denotes its eigenvalues by $\lambda_1, \dots, \lambda_n$. Then*

$$\det A = \lambda_1 \lambda_2 \cdots \lambda_n$$

*and*

$$\mathrm{Tr} A := \sum_{j=1}^{n} a_{ii} = \sum_{j=1}^{n} \lambda_j \ .$$

*Proof.* The first relation follows from

$$\det A = \det V T V^{-1} = \det T = \lambda_1 \lambda_2 \cdots \lambda_n \ .$$

The second relation follows from the fact that for any two $n \times n$ matrices $A, B$,

$$\mathrm{Tr} AB = \mathrm{Tr} BA$$

which is very easy to see. Now

$$\mathrm{Tr} A = \mathrm{Tr} V T V^{-1} = \mathrm{Tr} V^{-1} V T = \mathrm{Tr} T \ .$$

This proves the theorem. □

We continue discussing Cayley's theorem. If $p(\lambda)$ is a polynomial of degree $n$ we may consider $p(A)$ where $A$ is an $n \times n$ matrix. The polynomial can be written as

$$p(\lambda) = \sum_{j=0}^{n} c_j \lambda^j = c_0 + c_1 \lambda + c_2 \lambda^2 + \cdots + c_n \lambda^n$$

and then we define

$$p(A) = c_0 I_n + c_1 A + c_2 A^2 + \cdots + c_n A^n \ .$$

Over the complex numbers we may factor $p(\lambda)$ and write

$$p(\lambda) = a(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

where $\lambda_1, \dots \lambda_n$ are the roots of the polynomial $p(\lambda)$ and $a$ is some constant. It is easy to see that

$$p(A) = a(A - \lambda_1 I_n)(A - \lambda_2 I_n) \cdots (A - \lambda_n I_n) \ .$$

**Theorem 3.3.** *Let $A$ be an $n \times n$ matrix and $p(\lambda) := \det(A - \lambda I_n)$ its characteristic polynomial. Then*

$$p(A) = 0$$

*i.e., $A$ is a 'root' of its characteristic polynomial.*

*False proof.* The following is nonsensical (why?)

$$p(A) = \det(A - A I_n) = \det(A - A) = \det 0 = 0$$

□

*Proof.* We write the characteristic polynomial in the form

$$p(\lambda) = c_0 + c_1 \lambda + \cdots + c_n \lambda^n$$

and note that

$$c_0 I_n + c_1 A + \cdots + c_n A^n = V(c_0 I_n + c_1 T + \cdots + c_n T^n) V^{-1}$$

and hence it suffices to prove the theorem for the upper triangular matrix $T$ which amounts to showing that

$$(T - \lambda_1 I_n)(T - \lambda_2 I_n) \cdots (T - \lambda_n I_n) = 0 \ .$$

The first column of the upper triangular matrix $(T - \lambda_1 I_n)$ consists only of zeros and the matrix $(T - \lambda_2 I_n)$ is upper triangular but the second entry of the second column is zero. The first two columns of the product $(T - \lambda_1 I_n)(T - \lambda_2 I_n)$ must therefore be zero. The same reasoning shows that $(T - \lambda_1 I_n)(T - \lambda_2 I_n)(T - \lambda_3 I_n)$ has the first three columns vanishing and so forth. This proves Cayley's theorem. □

An interesting aspect of Cayley's theorem is that the first $n$ powers of an $n \times n$ matrix are linearly dependent. Thus, *any* power of $A$ can be written as a linear combination of $I_n, A, \ldots, A^{n-1}$, which is somewhat surprising. **Example:** Consider the Fibonacci matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

whose characteristic polynomial is $\lambda^2 - \lambda - 1$. Hence we have that

$$A^2 = I + A$$

So

$$A^3 = A + A^2 = A + I + A = 2A + I$$
$$A^4 = 2A^2 + A = 2(A + I) + A = 3A + 2I$$

From which we glean the structure

$$A^n = a_n A + a_{n-1} I \ .$$

Indeed,

$$A^{n+1} = a_n A^2 + a_{n-1} A = (a_n + a_{n-1})A + a_n I$$

and we get the recursion $a_{n+1} = a_n + a_{n-1}$ which is the Fibonacci sequence. Of course, while this is interesting from a theoretical point of view, it is less useful in practice.

## 4. Hermitean matrices

A matrix $A$ is hermitean if $A^* = A$. This is one the important classes of matrices chiefly because the ubiquitous in applications, like quantum mechanics and the can be diagonalized. Recall from (1) that for a hermitean matrix $\langle \vec{z}, A\vec{w} \rangle = \langle A\vec{z}, \vec{w} \rangle$.

**Lemma 4.1.** *The eigenvalues of a hermitean matrix are real.*

*Proof.* Suppose $\lambda$ is an eigenvalue of $A$ with eigenvector $\vec{w}$. Then

$$\lambda \langle \vec{w}, \vec{w} \rangle = \langle \vec{w}, \lambda \vec{w} \rangle = \langle \vec{w}, A\vec{w} \rangle \ .$$

A similar computation shows that

$$\overline{\lambda} \langle \vec{w}, \vec{w} \rangle = \langle \vec{A} w, \vec{w} \rangle = \langle \vec{w}, A^* \vec{w} \rangle$$

and since $A = A^*$ it follows that $\lambda \langle \vec{w}, \vec{w} \rangle = \overline{\lambda} \langle \vec{w}, \vec{w} \rangle$ and the eigenvalue is real. □

The notion of invariant subspace is a useful one.

**Definition 4.2.** *A subspace $V \subset \mathbb{C}^n$ is invariant under $A$, if for every $\vec{w} \in V$, $A\vec{w} \in V$.*

Another key fact about hermitean matrices is the following

**Lemma 4.3.** *Let $V \subset \mathbb{C}^n$ be a subspace invariant under $A$. The its orthogonal complement $V^\perp$ is also invariant under $A$.*

*Proof.* Pick any $\vec{z} \in V^\perp$ and any $\vec{w} \in V$. Then

$$\langle \vec{w}, A\vec{z} \rangle = \langle A\vec{w}, \vec{z} \rangle = 0$$

since $A\vec{w} \in V$. Since $\vec{w} \in V$ is arbitrary, $A\vec{z} \in V^\perp$. □

Here is a statement that shows why the invariant subspace notion is useful.

**Lemma 4.4.** *Let $A$ be any complex $n \times n$ matrix and assume that $V$ is invariant under $A$. Set $k = \dim V$. There exists a unitary matrix $U$ such that*

$$U^* A U = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

*where $B$ is a $k \times k$ matrix, $C$ a $k \times n - k$ matrix and $D$ a $(n-k) \times (n-k)$ matrix. The zero matrix at the bottom is a $(n-k) \times k$ matrix. Note that $V^\perp$ is not invariant under $A$, because we do not assume that $A = A^*$.*

*Proof.* Pick an orthonormal basis $\vec{u}_1, \ldots, \vec{u}_k$ in $V$ and an orthonormal basis $\vec{u}_{k+1}, \ldots, \vec{u}_n$ in $V^\perp$. Since $V$ is invariant under $A$ we have for $1 \le j \le k$

$$A\vec{u}_j = \sum_{\ell=1^k} B_{j\ell} \vec{u}_\ell$$

for some coefficients $B_{j\ell}$. If $k+1 \le j \le n$ we have that

$$A\vec{u}_j = \sum_{\ell=1}^{k} C_{j\ell} \vec{u}_\ell + \sum_{\ell=k+1}^{n} D_{j\ell} \vec{u}_\ell \ .$$

Form the matrix

$$U = [\vec{u}_1, \ldots, \vec{u}_n] \ ,$$

which is a unitary matrix, and note that

$$AU = [A\vec{u}_1, \ldots, A\vec{u}_k, A\vec{u}_{k+1} \vec{u}_n] = [\vec{u}_1, \ldots, \vec{u}_n] \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} = U \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \ ,$$

which proves the claim. □

As a corollary we have that

**Theorem 4.5.** *Let $A$ be a hermitean $n \times n$ matrix and $V \subset \mathbb{C}^n$ a $k$-dimensional subspace invariant under $A$. Then there exists a unitary matrix $U$ such that $U^*AU$ has the form*

$$U^* A U = \begin{bmatrix} B & 0 \\ 0 & D \end{bmatrix} \ .$$

*The $k \times k$ matrix $B$ and the $(n-k) \times (n-k)$ matrix $D$ are both hermitean.*

*Proof.* Both subspaces $V$ and $V^\perp$ are invariant under $A$ and hence the result follows from the previous lemma.

□