STANDARD FORMULA FOR DETERMINANTS

Imagine a set of n distinct objects. In order to distinguish them we give each one of them a label. A permutation is now a relabeling. Alternatively, consider the set $S = \{1, 2, \dots, n\}$. A permutation $\pi : S \to S$ is a function which one to one and hence onto (why?). We can express permutations in the following way

$$\begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ \pi(1) & \pi(2) & \dots & \pi(n-1) & \pi(n) \end{pmatrix}$$

E.g.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

corresponds to the function $\pi(1) = 2, \pi(2) = 4, \pi(3) = 1$ and $\pi(4) = 3$. The set of permutations of n objects is called the symmetric group and is denoted by $\mathcal{S}_n$.

Note that this function is one to one. Let $\pi \in \mathcal{S}_n$. Associated with this is a matrix P_π which is the matrix that emerges from the identity by permuting the columns using the permutation π. On has

Theorem 0.1. *Consider an $n \times n$ matrix A real or complex and denote the matrix elements by a_{ij} where both i and j vary between 1 and n. Then*

$$\det A = \sum_{\pi \in \mathcal{S}_n} (\det P_\pi) a_{1\pi(1)} \cdots a_{n\pi(n)}$$

In order to show this we call the right side of this equation $f(A)$. If $A = I$ we see that the only non-zero product appears when π is the identity permutation and hence $f(I) = 1$ since $\det I = 1$. Let A' be the matrix derived from A by exchanging two rows. We have to show that $f(A') = -f(A)$. To see this we observe that

$$a'_{1\pi(1)} a'_{2\pi(2)} \cdots a'_{n\pi(n)} = a_{2\pi(1)} a_{1\pi(2)} \cdots a_{n\pi(n)} = a_{1\pi(2)} a_{2\pi(1)} \cdots a_{n\pi(n)}$$

Denote by σ the permutation that exchanges 1 and 2 and leaves all others fixed. Thus, we may write

$$a'_{1\pi(1)} a'_{2\pi(2)} \cdots a'_{n\pi(n)} = a_{1\pi(\sigma(1))} a_{2\pi(\sigma(2))} \cdots a_{n\pi(\sigma(n))} = a_{\pi\circ\sigma(1)} \cdots a_{\pi\circ\sigma(n)}$$

Further

$$P_{\pi\circ\sigma} = P_\pi P_\sigma$$

and so

$$\det P_{\pi\circ\sigma} = \det P_\pi \det P_\sigma = -\det P_\pi \ .$$

Thus

$$f(A') = -\sum_{\pi \in \mathcal{S}_n} \det P_{\pi\circ\sigma} a_{\pi\circ\sigma(1)} \cdots a_{\pi\circ\sigma(n)}$$

and since $\pi \circ \sigma$ runs through all the permutations precisely once we have that

$$f(A') = -\sum_{\pi \in \mathcal{S}_n} \det P_\pi a_{\pi(1)} \cdots a_{\pi(n)} = -f(A)$$

That $f(A)$ is linear in each row is obvious. We have seen in the lecture that the these three properties determine $f(A)$ uniquely and hence $f(A) = \det A$. From this formula we can recover our previous results concerning the Trace of a matrix and its determinant. Consider

$$\det(A - \Lambda I) = \sum_{\pi \in \mathcal{S}_n} \det P_\pi (a_{1\pi(1)} - \lambda \delta_{1\pi(1)}) \cdots (a_{n\pi(n)} - \lambda \delta_{n\pi(n)})$$

Expanding this expression in powers of λ we see that the constant, i.e., the term independent of λ equals $\det A$. The term proportional to λ is given by

$$-\lambda \sum_{\pi \in \mathcal{S}_n} \det P_\pi \sum_{i=1}^n \delta_{1\pi(1)} \cdots \delta_{(i-1)\pi(i-1)} a_{i\pi(i)} \delta_{(i+1)\pi(i+1)} \cdots \delta_{n\pi(n)} \ .$$

This term is non-zero only if π is the identity permutation and hence this term equals

$$-\lambda \sum_{i=1}^n a_{ii} = -\lambda \mathrm{Tr} A \ .$$

Recall that the characteristic polynomial can be factored

$$\det(A - \lambda I) = (-1)^n (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$$

and by expanding we get that the constant term, i.e., the term independent of λ is given by $\lambda_1 \cdots \lambda_n$ and the term proportional to λ is given by

$$-\lambda \sum_{i=1}^n \lambda_i \ .$$

Thus we recover the identities

$$\lambda_1 \cdots \lambda_n = \det A$$

and

$$\sum_{i=1}^n \lambda_i = \mathrm{Tr} A \ .$$