THE DISCRETE FOURIER TRANSFORM

1. Roots of 1

First a little review about complex numbers, namely roots of 1. You know of course that the equation $x^2 = 1$ has two roots, $+1$ and -1. If we consider the equation $x^4 = 1$ and look for solution in the complex domain we find the four roots $1, i, -1, -i$ where you recall that $i^2 = -1$. Thus we can factor

$$x^4 - 1 = (x-1)(x-i)(x+1)(x+i) \ .$$

The roots of the equation $x^3 - 1 = 0$ are given by $1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$. and hence

$$x^3 - 1 = (x-1)(x - \frac{-1+i\sqrt{3}}{2})(x + \frac{1+i\sqrt{3}}{2}) \ .$$

Things become much clearer if we jot down these points in the complex plane. Th roots of the equation $x^4 - 1 = 0$ are on the unit circle and are the corners of a square and the roots $x^3 - 1 = 0$ are also on the unit circle and are the corners of an equilateral triangle.

Using a bit of trigonometry we find that the roots of $x^4 - 1 = 0$ can be written as

$$1 = \cos 0 + i \sin 0 = e^{i0} \ , \ i = \cos(\pi/2) + i\sin(\pi/2) = e^{i\pi/2} \ ,$$

$$-1 = \cos(\pi) + i\sin(\pi) = e^{i\pi} \ , \ -i = \cos(3\pi/2) + i\sin(3\pi/2) = e^{i3\pi/2} \ .$$

and likewise, the roots of the equation $x^3 - 1 = 0$ can be written as

$$1 = \cos 0 + i \sin 0 = e^{i0} \ , \frac{-1+i\sqrt{3}}{2} = \cos(2\pi/3) + i\sin(2\pi/3) = e^{i2\pi/3} \ ,$$

$$\frac{-1-i\sqrt{3}}{2} = \cos(4\pi/3) + i\sin(4\pi/3) = e^{i4\pi/3} \ .$$

For the general equation $x^n - 1 = 0$ we get the roots

$$1 \ , \ e^{2\pi i \frac{1}{n}} \ , \ e^{2\pi i \frac{2}{n}} \ , \ e^{2\pi i \frac{3}{n}} \ , \ \dots \ , e^{2\pi i \frac{n-1}{n}} \ .$$

To abbreviate the notation we set

$$\omega_n = e^{2\pi i \frac{1}{n}}$$

and can write the set of roots as

$$K' = \{1 \ , \ \omega_n \ , \ \omega_n^2 \ , \ \omega_n^3 \dots \ , \ \omega_n^{n-1}\} \ .$$

The following observation is important for what follows: If we multiply each element of K' by ω_n we get the same set back. In fact multiplying each element of K' in the order given above by ω_n permutes these elements cyclically.

2. The permutation matrix T

The $n \times n$ matrix T is is the matrix that maps the vector $[x_1, x_2, \ldots, x_n]$ to the vector $[x_n, x_1, x_2, \ldots, x_{n-1}]$. The matrix T can be written as

$$T = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}$$

it is clear that $T^n = I$. To compute the eigenvalues of T we have several avenues. One is to compute the characteristic polynomial, which is a bit tedious, but there is a much better way by systematically 'guessing' the eigenvectors. One obvious one is the vector consisting of 1s. Let us call it $\vec{v}_0$. Another, which gets us closer to the idea is the vector, call it $\vec{v}_1$, whose entries are $1, \omega_n, \omega_n^{,} \ldots, \omega_n^{n-1}$. Note that the vector $T\vec{v}_1$ consists of the vector whose entries are $\omega_n^{n-1}, 1, \omega_n, \ldots, \omega_n^{n-2}$. If we recall that $\omega_n^{n-1} = \omega_n^{-1}$ we get that $T\vec{v}_1 = \omega_n^{-1}\vec{v}_1$. Hence we see that $\vec{v}_1$ is another eigenvector. It is complex and obviously linearly independent from the vector $\vec{v}_0$. Let us pause for the moment and look at the structure of these two vectors. The second had entries that are powers of the root of 1 given by ω_n. The first vector is similar it is can be thought of as consisting of powers of another root of 1, namely 1. Hence we may continue and consider the vector $\vec{v}_2$ whose entries are given by the powers of ω_n^2. That is the vector is

$$\vec{v}_2 = \begin{bmatrix} 1 \\ \omega_n^2 \\ (\omega_n^2)^2 \\ (\omega_n^2)^3 \\ . \\ . \\ (\omega_n^2)^{n-1} \end{bmatrix} .$$

Once more

$$T\vec{v}_2 = \begin{bmatrix} (\omega_n^2)^{n-1} \\ 1 \\ \omega_n^2 \\ (\omega_n^2)^2 \\ (\omega_n^2)^3 \\ . \\ . \\ (\omega_n^2)^{n-2} \end{bmatrix} = (\omega_n^2)^{n-1} \begin{bmatrix} 1 \\ \omega_n^2 \\ (\omega_n^2)^2 \\ (\omega_n^2)^3 \\ . \\ . \\ (\omega_n^2)^{n-1} \end{bmatrix} = (\omega_n^2)^{-1}\vec{v}_2 .$$

Continuing this way we find that the eigenvalues of T are given by $1, \omega_n, \omega_n^2, \cdots, \omega_n^{n-1}$ and the eigenvectors arranged into a matrix are given by

$$F_n = \begin{bmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 & \ldots & \omega_n^{n-1} \\ 1 & \omega_n^2 & (\omega_n^2)^2 & (\omega_n^3)^2 & \ldots & (\omega_n^{n-1})^2 \\ . & . & . & . & \ldots & . \\ 1 & \omega_n^{n-1} & (\omega_n^2)^{n-1} & (\omega_n^3)^{n-1} & \ldots & (\omega_n^{n-1})^{n-1} \end{bmatrix} ,$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \omega_n^6 & \dots & \omega_n^{2(n-1)} \\ . & . & . & . & \dots & . \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \omega_n^{3(n-1)} & \dots & \omega_n^{(n-1)^2} \end{bmatrix}$$

We call this the **Fourier matrix**. Lets work all this out when $n = 4$. The roots are, as we have seen, $1, i, i^2, i^3, i^4$ or $1, i, -1, -i$. Then the matrix of eigenvectors is given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & (i^2) & (i^2)^2 & (i^2)^3 \\ 1 & (i^3) & (i^3)^2 & (i^3)^3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

Note that the column vectors of this matrix are orthogonal with respect to the inner product and hence, normalizing these vectors, we get

$$U = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} , \tag{1}$$

a unitary matrix. One computes easily that $TU = UD$ where

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix}$$

3. Why discrete Fourier transform

We stay with the four dimensional situation we talked about t=in the previous section. We have seen that

$$T = UDU^*$$

and one easily computes that

$$U^* = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} ,$$

noting that $U^T = U$. The four eigenvectors which are the column vectors in U we denoted by $\vec{v}_0, \vec{v}_1, \vec{v}_2, \vec{v}_3$. They are orthonormal with respect to the inner product

$$\langle \vec{z}, \vec{w} \rangle = \sum \bar{z}_i w_i \ .$$

Given and vector $\vec{x}$ real or otherwise, we compute the **Fourier Coefficients**

$$\langle \vec{v}_0, \vec{x} \rangle \ , \ \langle \vec{v}_1, \vec{x} \rangle \ , \langle \vec{v}_2, \vec{x} \rangle \ , \ \langle \vec{v}_3, \vec{x} \rangle$$

then

$$\vec{x} = \langle \vec{v}_0, \vec{x} \rangle \vec{v}_0 + \langle \vec{v}_1, \vec{x} \rangle \vec{v}_1 + \langle \vec{v}_2, \vec{x} \rangle \vec{v}_2 + \langle \vec{v}_3, \vec{x} \rangle \vec{v}_3$$

then

$$T\vec{x} = \langle \vec{v}_0, \vec{x} \rangle \vec{v}_0 - i \langle \vec{v}_1, \vec{x} \rangle \vec{v}_1 - \langle \vec{v}_2, \vec{x} \rangle \vec{v}_2 + i \langle \vec{v}_3, \vec{x} \rangle \vec{v}_3 \ .$$

4. Fast Fourier transform

The Fourier matrix is not sparse and hence to compute $F_n\vec{x}$ it takes about n^2 operations. We shall show by arranging the computation in a clever way, that it takes much fewer steps. To describe the result we set $n = 2^k$.

Theorem 4.1. *Let $\omega_n = e^{\frac{2\pi i}{n}}$ with $n = 2^k$. One can calculate $F_n\vec{x}$ for any vector $\vec{x} \in \mathbb{C}^n$ in $4 \cdot 2^k k = 4n\log_2(n)$ operations.*

Consider the case where $n = 2m$. Given an arbitrary n vector $\vec{x}$. Write it in the form $\vec{x} = \vec{x}_0$ and $\vec{x}_1$ where $\vec{x}_0$ contains only the entries of $\vec{x}$ with even index and likewise $\vec{x}_1$ the entries with odd index. It we write $\vec{y}_0 = [x_0, x_2, \ldots, x_{2(m-1)}]$ and $\vec{y}_1 = [x_1, x_3, \ldots, x_{2m-1}]$ then we can write the vector

$$\begin{bmatrix} \vec{y}_0 \\ \vec{y}_1 \end{bmatrix} = P\vec{x}$$

where P is the permutation matrix that maps the indices $(0, 2, \ldots, 2(m-1))$ to $(0, 1, \ldots, m-1)$ and the indices $(1, 2, \ldots 2m-1)$ to the indices $(m, \ldots, 2m-1)$. The point now is that

$$[F_{2m}\vec{x}]_j = \sum_{\ell=0}^{2m-1} \omega_{2m}^{j\ell} x_\ell = \sum_{\ell=0}^{m-1} \omega_{2m}^{j2\ell} x_{2\ell} + \sum_{\ell=0}^{m-1} \omega_{2m}^{j(2\ell+1)} x_{2\ell+1}$$

$$= \sum_{\ell=0}^{m-1} \omega_m^{j\ell} x_{2\ell} + \omega_{2m}^{j} \sum_{\ell=0}^{m-1} \omega_m^{j\ell} x_{2\ell+1}$$

using that

$$\omega_{2m}^{2j\ell} = e^{\frac{2\pi i 2l\ell}{2m}} = e^{\frac{2\pi i l\ell}{m}} = \omega_m^{j\ell} .$$

We can rewrite this using the vectors $\vec{y}_0$ and $\vec{y}_1$ (which are m vector) as

$$[F_{2m}\vec{x}]_j = [F_m\vec{y}_0]_j + \omega_{2m}^{j}[F_m\vec{y}_1]_j .$$

As matrices we can write this as

$$\begin{bmatrix} I & D_m \\ I & -D_m \end{bmatrix} \begin{bmatrix} F_m & 0 \\ 0 & F_m \end{bmatrix}$$

where D_m is the diagonal matrix with the elements $1, \omega_m, \omega_m^2, \ldots, \omega_m^{m-1}$ on the diagonal. These entries cover the indices $j = 0, \ldots, m-1$. The when $j \geq m$, then $\omega_{2m}^{j} = -\omega_{2m}^{j-m}$ and hence the negative sign in front of D_m in the second row. Hence we have that

$$F_{2m} = \begin{bmatrix} I & D_m \\ I & -D_m \end{bmatrix} \begin{bmatrix} F_m & 0 \\ 0 & F_m \end{bmatrix} P$$

Computing $P\vec{x}$ does not use any operations, we just group the even indexed and odd indexed elements together. This is achieved by a suitable input routine. Let $c(m)$ be the smallest number of steps to compute $F_m\vec{y}$. That gives us $2c(m)$ steps to compute

$$\begin{bmatrix} F_m & 0 \\ 0 & F_m \end{bmatrix} P\vec{x}$$

To compute $D_m F_m$ that takes another m steps, because D_m is diagonal. Hence we need $2c(m) + m$ steps to compute $F_{2m}\vec{x}$. In other words, if $c(m)$ denotes the smallest number of steps to compute $F_m\vec{y}$ then

$$c(2m) \leq 2c(m) + m .$$

Suppose we pick $n = 2^k$ then $c(2^k) \leq 2c(2^{k-1}) + 2^{k-1}$ This leads to a recursion which can be solved with the result that

$$c(2^k) \leq 2^k a_0 + k2^{k-1}$$

Here a_0 is the number of steps to compute the Fourier transform for a two vector which takes two steps. Thus, if we stick n back in, we get that the multiplication of the Fourier matrix F_n with an arbitrary vector takes

$$n(a_0 + \frac{1}{2}\log_2 n)$$

step. If we choose $n = 2^{20}$ which about a million by million matrix, it takes about $10^6(a_0+10)$ which should be compared with the naive computation which would give 10^{12}.

5. Application to differential equations

Consider the system

$$\frac{d^2x_i}{dt^2}_i = \omega^2(x_{i-1} - 2x_i + x_{i+1})$$

where $i = 1, \ldots, N$ with the convention that $N + 1 \equiv 1$. If we write this in vector form we get that

$$\frac{d^2}{dt^2}\vec{X} = \omega^2[T - 2I + T^{-1}]\vec{X}$$

As an example, take $N = 4$. Then we get the equations

$$\frac{d^2x_1}{dt^2}_i = \omega^2(x_4 - 2x_1 + x_2)\ ,$$
$$\frac{d^2x_2}{dt^2}_i = \omega^2(x_1 - 2x_2 + x_3)\ ,$$
$$\frac{d^2x_3}{dt^2}_i = \omega^2(x_2 - 2x_3 + x_4)\ ,$$
$$\frac{d^2x_4}{dt^2}_i = \omega^2(x_3 - 2x_4 + x_1)\ .$$

We have diagonalized T. To stay with this example, we get the eigenvalues