PRACTICE FINAL EXAM

1. Linear systems of equation

Problem 1: Find the inverse matrix of

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}.$$

Problem 2: Compute L and U for the symmetric matrix

$$\begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots.

Problem 3: Consider the subspace of $\mathbb{R}^4$ that given by the equation

$$w + x + y + z = 0$$

Find a basis for this subspace. What is its dimension?

2. Orthogonality

Problem 4: Consider the matrix

$$\begin{bmatrix} 1 & 0 & 2 & -3 \\ 2 & 6 & -2 & 12 \\ 2 & 3 & 1 & 3 \end{bmatrix}$$

a) Find a basis for the column space $C(A)$
b) Find a basis for $N(A)$
c) For $C(A^T)$
d) For $N(A^T)$.

Problem 5: Find an orthonormal basis for the subspace of Problem 3.

Problem 6: Consider the two lines in $\mathbb{R}^4$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Find the distance vector, i.e., between them. Compute its length. (Hint: Formulate this as a least square problem)

Problem 7: Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = 1$, $b = 7$ at $t = -1$ and $b = 21$ at $t = 2$. Find the least square solution $\widehat{x} = (C, D)$.

Problem 8: Find the QR factorization of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

and compute the projection of the vector

$$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

onto the column space of A

3. Eigenvalues and eigenvectors

Problem 9: A two by two matrix A satisfies the matrix equation

$$A^2 - 5A + 6I = 0 \ .$$

What are the eigenvalues of the matrix? Is it diagonalizable?

Problem 10: Compute $\lim_{k\to\infty} P^k$ where

$$P = \begin{bmatrix} \frac{1}{10} & \frac{5}{10} \\ \frac{9}{10} & \frac{5}{10} \end{bmatrix}$$

Problem 11: Find a singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Problem 12: Prove or find a counterexample:

a) A set of mutually orthogonal vectors is always linearly independent.

b) If A is an $m \times n$ matrix with linear independent columns, then $A^T A$ as invertible.

c) If A is an $m \times n$ matrix with linear independent columns, then AA^T as invertible.

d) If A is any $m \times n$ matrix, then A and A^T have the same non-zero singular values.

e) If A and B are both $n \times n$ matrices the AB and BA have the same eigenvalues.