PRACTICE FINAL EXAM

1. Linear systems of equation

Problem 1: Find the inverse matrix of

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}.$$

Solution: Augmented matrix:

$$A = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right].$$

Row reducing to reduced echelon form yields the interesting result

$$A = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 3 & -3 & 1 \end{array}\right].$$

Problem 2: Compute L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots.

Solution: First consider the augmented matrix $[A|I]$. Using row reduction this matrix reduces to

$$[L^{-1}A|L^{-1}] = [U|L^{-1}]$$

Hence all we have to do is invert L^{-1}. Performing this row reduction yields

$$U = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

and

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

which can easily be inverted and yields

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

There are four pivots if and only if $a \neq 0$ and $a \neq b$, $b \neq c$ and $d \neq c$. Now in this case there is another of computing L. Just compute $L = AU^{-1}$. The inverse of U one can get through back substitution.

Problem 3: Consider the subspace of $\mathbb{R}^4$ that given by the equation

$$w + x + y + z = 0$$

Find a basis for this subspace. What is its dimension?

Solution: Row reduction is here trivial and we have one pivot and three free variables x, y, z. Hence we have the general solution

$$\begin{bmatrix} -x-y-z \\ x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence the vectors

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

form a basis of this three dimensional space.

2. Orthogonality

Problem 4: Consider the matrix

$$\begin{bmatrix} 1 & 0 & 2 & -3 \\ 2 & 6 & -2 & 12 \\ 2 & 3 & 1 & 3 \end{bmatrix}$$

a) Find a basis for the column space $C(A)$
b) Find a basis for $N(A)$
c) For $C(A^T)$
d) For $N(A^T)$.

Solution: Row reduction leads to following reduced echelon form

$$\begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first two columns are pivot columns and hence

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}$$

is a basis for $C(A)$. The row space does not change and hence

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

is a basis for $C(A^T)$. The third and fourth variables are free and hence

$$\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

is a basis for $N(A)$. The $N(A^T)$ is the orthogonal complement of $C(A)$ and hence

$$\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

is a basis for $N(A^T)$.

Problem 5: Find an orthonormal basis for the subspace of Problem 3.

Solution: We use the Gram Schmidt method.

$$\vec{A} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then

$$\vec{B} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$$

Next

$$\vec{C} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}.$$

Hence we have the orthonormal basis

$$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \vec{v}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \ \vec{v}_3 = \frac{1}{2\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}$$

Problem 6: Consider the two lines in $\mathbb{R}^4$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Find the distance vector, i.e., between them. Compute its length. (Hint: Formulate this as a least square problem)

Solution: We have to choose s, t so that

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad - \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - t \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

has minimal length which is the same as minimizing the length of

$$\begin{bmatrix} 1 & -1 \\ 1 & -2 \\ 1 & -3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \quad - \quad \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} .$$

We set

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -2 \\ 1 & -3 \\ 1 & -4 \end{bmatrix}, \vec{b} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{x} = \begin{bmatrix} s \\ t \end{bmatrix} .$$

The normal equations are $A^T A \vec{x} = A^T \vec{b}$ or

$$\begin{bmatrix} 4 & -10 \\ -10 & 30 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

which yields $s = -\frac{1}{2}, t = -\frac{1}{5}$. The points of minimal distance on the lines are

$$\frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \quad \text{and} \quad \frac{1}{5} \begin{bmatrix} -1 \\ 3 \\ -3 \\ -4 \end{bmatrix}$$

and the difference vector is

$$\frac{1}{10}\begin{bmatrix} 7 \\ -11 \\ 1 \\ 3 \end{bmatrix}$$

which, as one can easily check, is perpendicular to both lines. The distance is the length of this vector:

$$\frac{1}{10}\sqrt{49+121+1+9} = \frac{3}{\sqrt{5}}$$

Problem 7: Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = 1$, $b = 7$ at $t = -1$ and $b = 21$ at $t = 2$. Find the least square solution $\widehat{x} = (C, D)$.

Solution: The vector of the t-values is

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

and the b values

$$\begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}.$$

If the data *would* fit a line then we could find C, D such that

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$$

There is no such solution and hence we solve the least square problem $A^T A\vec{x} = A^T\vec{b}$

$$\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$$

or

$$C = 9, D = 4$$

The best linear fit is thus given by the line

$$b = 9 + 4t$$

Problem 8: Find the QR factorization of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

and compute the projection of the vector

$$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

onto the column space of A

Solution: Using the Gram-Schmidt method we have

$$\vec{q}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

A vector in the column space that is perpendicular to $\vec{q}_1$ is given by

$$\vec{B} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

Hence

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{-3}{2\sqrt{5}} \\ \frac{1}{2} & \frac{-1}{2\sqrt{5}} \\ \frac{1}{2} & \frac{1}{2\sqrt{5}} \\ \frac{1}{2} & \frac{3}{2\sqrt{5}} \end{bmatrix}$$

and

$$R = Q^T A = \begin{bmatrix} 2 & 5 \\ 0 & \sqrt{5} \end{bmatrix}.$$

Hence

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-3}{2\sqrt{5}} \\ \frac{1}{2} & \frac{-1}{2\sqrt{5}} \\ \frac{1}{2} & \frac{1}{2\sqrt{5}} \\ \frac{1}{2} & \frac{3}{2\sqrt{5}} \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 0 & \sqrt{5} \end{bmatrix}$$

Recall that $Q^TQ = I$ but QQ^T is the projection onto the subspace spanned by the column vectors of Q. Hence

$$QQ^T\vec{b} = \begin{bmatrix} \frac{1}{2} & \frac{-3}{2\sqrt{5}} \\ \frac{1}{2} & \frac{-1}{2\sqrt{5}} \\ \frac{1}{2} & \frac{1}{2\sqrt{5}} \\ \frac{1}{2} & \frac{3}{2\sqrt{5}} \end{bmatrix}\begin{bmatrix} 1 \\ -\frac{1}{\sqrt{5}} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

One can easily check that

$$\vec{b} - QQ^T\vec{b}$$

is perpendicular to the column vectors of Q.

3. Eigenvalues and eigenvectors

Problem 9: A two by two matrix A satisfies the matrix equation

$$A^2 - 5A + 6I = 0 \ .$$

What are the eigenvalues of the matrix? Is it diagonalizable?

Solution: We can write

$$A^2 - 5A + 6I = (A - 2I)(A - 3I) = 0 \ ,$$

and hence, if λ is an eigenvalue of A, then it must satisfy the equation

$$\lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2) = 0 \ .$$

Thus, A could have the following eigenvalues: $2, 3$, a double eigenvalue $2, 2$, or a double eigenvalue $3, 3$. Assume that A has 2 as a double eigenvalue. Then 3 is not an eigenvalue and hence $A - 3I$ is invertible. Thus $0 = (A - 2I)(A - 3I)$ implies that $A - 2I = 0$ or $A = 2I$. The same argument shows that if A has 3 as a double eigenvalue, then $A = 3I$. The other possibility is the A has both 2 and 3 as eigenvalues. In all these cases, A can be diagonalized.

Problem 10: Compute $\lim_{k\to\infty} P^k$ where

$$P = \begin{bmatrix} \frac{1}{10} & \frac{5}{10} \\ \frac{9}{10} & \frac{5}{10} \end{bmatrix}$$

Solution: First we find the eigenvalues and eigenvectors for P. One eigenvalue is 1 which is easy because the matrix is stochastic. The corresponding eigenvector is

$$\vec{v}_1 = \frac{1}{14} \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

Note that I normalized the vector so that the components are probabilities. The other eigenvalue is $-4/10$. This follows from the fact that the trace of the matrix P is $6/10$ which must be the sum of the eigenvalues. The other eigenvector is

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \ .$$

Now form

$$V = \begin{bmatrix} \frac{5}{14} & 1 \\ \frac{9}{14} & -1 \end{bmatrix} \text{ so that } V^{-1} = \begin{bmatrix} 1 & 1 \\ \frac{9}{14} & -\frac{5}{14} \end{bmatrix}$$

Now

$$P^k = V \begin{bmatrix} 1 & 0 \\ 0 & (-\frac{4}{10})^k \end{bmatrix} V^{-1}$$

which, as $k \to \infty$, converges to

$$V \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} V^{-1} = \begin{bmatrix} \frac{5}{14} & 1 \\ \frac{9}{14} & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{9}{14} & -\frac{5}{14} \end{bmatrix} = \begin{bmatrix} \frac{5}{14} & \frac{5}{14} \\ \frac{9}{14} & \frac{9}{14} \end{bmatrix}$$

Here is another argument without computing the second eigenvector. P is diagonalizable and hence

$$P = VDV^{-1} \text{ and therefore } P^k = VD^kV^{-1}$$

where D is diagonal. As $k \to \infty$ only the eigenvalue 1 survives and we have that

$$\lim_{k\to\infty} P^k = V \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} V^{-1}$$

Now

$$V = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}$$

where $\vec{w}$ is the second eigenvector. Hence

$$V \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \vec{v} & 0 \end{bmatrix}$$

Now write

$$V^{-1} = \begin{bmatrix} \vec{a}^T \\ \vec{b}^T \end{bmatrix}$$

so that

$$V \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} V^{-1} = \vec{v}\vec{a}^T \ .$$

Thus

$$\lim_{k\to\infty} P^k = \vec{v}\vec{a}^T$$

Now we don't know $\vec{a}$. Note, however, that P is stochastic and hence $\lim_{k\to\infty} P^k$ is also stochastic. Stochastic means that

$$[1,1]P = [1,1]$$

and hence

$$[1,1] = [1,1]\vec{v}\vec{a}^T = \vec{a}^T$$

since $\vec{v}$ is a probability vector. Hence

$$\lim_{k\to\infty} P^k = \frac{1}{14} \begin{bmatrix} 5 \\ 9 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 5 & 5 \\ 9 & 9 \end{bmatrix}$$

It is interesting that this argument works for general $n \times n$ stochastic matrices as long as the the other eigenvalues in magnitude are strictly smaller than 1. All we need is the eigenvector $\vec{v}$ with $P\vec{v} = \vec{v}$, $\vec{v}$ a probability vector, and then

$$\lim_{k\to\infty} P^k = \begin{bmatrix} \vec{v} & \vec{v} & \cdots & \vec{v} \end{bmatrix}$$

Problem 11: Find a singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Solution: First we have to compute either A^TA or AA^T. The first yields a 3×3 matrix whereas the second yields a 2×2 matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

which is easier to deal with. The normalized eigenvectors are

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and the corresponding eigenvalues are 3 and 1.The singular values are $\sqrt{3}$ and 1. Now one has to remember the order of the matrices. The way I do it is to think of the singular value decomposition in the form

$$A = V\Sigma U^T \ .$$

This means that the matrix we computed

$$AA^T = V\Sigma U^T U \Sigma V^T = V\Sigma^2 V^T \ .$$

Hence

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix} \ , \ V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

To find U we compute

$$U^T = \Sigma^{-1}V^T A = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 2 & 1 \\ -\sqrt{3} & 0 & \sqrt{3} \end{bmatrix}$$

Hence

$$A = V\Sigma U^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 2 & 1 \\ -\sqrt{3} & 0 & \sqrt{3} \end{bmatrix}$$

Problem 12: True or False:

a) A set of mutually orthogonal vectors is always linearly independent. TRUE

To see this take the orthogonal vectors $\vec{v}_1, \dots, \vec{v}_n$. One has to show that

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = 0$$

implies that $c_1 = c_2 = \cdots = c_n = 0$. Take the dot product with $\vec{v}_1$ and we get that $c_1\vec{v}_1 \cdot \vec{v}_1 = 0$ all the other dot products vanish. $\vec{v}_1$ should not be zero (I forgot to write that assumption). Hence $c_1 = 0$. Now repeat the argument with $\vec{v}_2$ etc.

b) If A is an $m \times n$ matrix with linear independent columns, then A^TA as invertible. TRUE

The matrix A and A^TA have the same null space and since the column vectors of A are independent we have that $N(A) = \{0\} = N(A^TA)$ and hence A^TA is invertible.

c) If A is an $m \times n$ matrix with linear independent columns, then AA^T as invertible. FALSE

Take the matrix

$$A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so that

$$AA^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which is not invertible.

d) If A is any $m \times n$ matrix, then A and A^T have the same non-zero singular values. TRUE

The matrix AA^T and A^TA have the same non-zero eigenvalues.

e) If A and B are both $n \times n$ matrices the AB and BA have the same eigenvalues. TRUE

They have the same non-zero eigenvalues. If $AB\vec{v} = \lambda\vec{v}$ and $\lambda \neq 0$, then $B\vec{v} \neq 0$ and hence

$$BA(B\vec{v}) = B(AB\vec{v}) = \lambda B\vec{v} \ .$$

AB has a zero eigenvalue if and only if $\det(AB) = \det(BA) = 0$, if and only of BA has a zero eigenvalue.