PRACTICE TEST 1, SOLUTIONS

Problem 1: a) By computing the row reduced echelon form find all the solutions of the system $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

Row reduction of the augmented matrix leads to

$$\begin{bmatrix} 1 & 0 & 2 & | & -3 \\ 0 & 1 & -3 & | & 2 \end{bmatrix}$$

The free variable is x_3 and we get the solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

b) Indicate the pivot columns. The first and second column in the matrix A are the pivot columns.

c) What is the rank of A? The rank is 2.

Problem 2: a) Find a 3×3 matrix E that when multiplied with a 3×3 matrix A adds three times the first *column* of A to the second *column* of A. (Hint: Think of AE and not EA.)

$$E = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so that

$$\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} E = \begin{bmatrix} \vec{a} & \vec{b} + 3\vec{a} & \vec{c} \end{bmatrix}$$

b) What is the inverse of E?

$$E = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 3: a) Are the three vectors below linearly independent?

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

We have to row reduce

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

which leads to

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Every column is a pivot column and hence the equation $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^3$. This means that the three vectors span $\mathbb{R}^3$. Likewise, there are no free variables and hence the equation $A\vec{x} = 0$ has only the zero solution and hence the vectors are linearly independent.

b) An $n \times n$ matrix is invertible if and only if the column vectors form a basis for $\mathbb{R}^n$. Explain this.

Given a basis $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ we know that the matrix $A = [\vec{v}_1, \ldots, \vec{v}_n]$ must have rank $r = n$. There are no free variables since the vectors are linearly independent and hence every column has a pivot. As a consequence the equation $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^n$. Thus, A has a right inverse. Likewise, we can always find a matrix C which is a product of row reduction moves such that CA is of reduced echelon form. Since every column and every row has a pivot this matrix is the identiy matrix. Thus, A is invertible.

Conversely, if A is invertible, the equation $A\vec{x} = 0$ can be solved by multiplying both sides by A^{-1} and get that $\vec{x} = 0$ is the only solution. Thus the vectors are linearly independent. Now the equation $A\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^n$ which is given by $A^{-1}\vec{b}$. Thus, the column vectors must be a spanning set and the vectors $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

Problem 4: Find the inverse of the matrix

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

I got

$$\begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

Problem 5: a) Find a 2×2 matrix whose column space and null space are equal.

The matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has this property. The column space consists of all multiple of the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and likewise does the null space.

b) Is the same true for a symmetric 2×2 matrix? Explain.

No, this is not true. Consider the matrix

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

In order to have a non-trivial null space we need that $ax + by = 0$, $bx + cy = 0$ has a non-zero solution. This means that the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ is perpendicular to the vectors $\begin{bmatrix} a \\ b \end{bmatrix}$

and $\begin{bmatrix} b \\ c \end{bmatrix}$, i.e., the vectors in the null space are perpendicular to the vectors in the column space. This holds in general only for symmetric matrices. Needless to say that in order to have a non-trivial null space we need that the vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} b \\ c \end{bmatrix}$ are proportional.

Problem 6: Find a basis for $N(A), C(A), N(A^T)$ and $C(A^T)$ where

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 2 & 6 & 6 \\ 2 & 3 & 0 \end{bmatrix}$$

Solution: Row reduction leads to the row reduced echelon form

$$R = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

the first two columns of A are pivot columns and hence

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}$$

are a basis for $C(A)$. The row space does not change under row reduction and hence the vectors

$$\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

are a basis for the $C(A^T)$. The last column of R corresponds to a free variables and solving the system for vectors in the null space of A yields

$$\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

as a basis for $N(A)$. The space $N(A^T)$ is the orthogonal complement of $C(A)$ and hence

$$\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

will be a basis for $N(A^T)$. Note, that we row reduced only once and simply used the relation ships of the various subspaces to find bases.

Problem 7: Using the normal equations, solve the least square problem for $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 2 & -1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

Solution: Recall that if $\vec{x}$ is the a least square solution then $A\vec{x} - \vec{b}$ must be perpendicular to the column space $C(A)$ and hence in the null space of A^T. Thus,

$$A^T A\vec{x} = A^T\vec{b}$$

are the normal equations. We compute

$$A^T A = \begin{bmatrix} 9 & 3 \\ 3 & 11 \end{bmatrix}, A^T\vec{b} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$\vec{x} = (A^T A)^{-1} A^T\vec{b} = \frac{1}{90}\begin{bmatrix} 11 & -3 \\ -3 & 9 \end{bmatrix}\begin{bmatrix} 6 \\ 6 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

Check:

$$A\vec{x} - \vec{b} = \frac{1}{15}\begin{bmatrix} 26 \\ 22 \\ 10 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} -4 \\ 7 \\ -5 \end{bmatrix}$$

which is indeed perpendicular to $C(A)$.

Problem 8: a) Find the QR factorization of the matrix

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 2 & 6 & 6 \\ 2 & 3 & 0 \end{bmatrix}$$

b) Using the result of a) find the least square solutions for the equation $A\vec{x}$ where $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Solution: We apply the Gram-Schmidt procedure:

$$\vec{A} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\vec{B} = \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

$$\vec{C} = \begin{bmatrix} -3 \\ 6 \\ 0 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} -3 \\ 6 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} -3 \\ 6 \\ 0 \end{bmatrix}\begin{bmatrix} -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 6 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

That one gets the zero vector is less of a surrpise if one realizes that the rank of A is 2. Now we form the matrix of normalized vectors

$$Q = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & 2 \\ 2 & -1 \end{bmatrix} .$$

Note that one could include the zero vector but that does not add anything essential, since

$$R = Q^T A = \begin{bmatrix} 3 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix}$$

Check that $QR = A$.

To solve b) recall that the column vectors of Q form an orthonormal basis of the column space $C(A)$. For the least square solution the vector $A\vec{x} - \vec{b}$ has to be perpendicular to $C(A)$, i.e., perpendicular to the column vectors of A. Hence

$$Q^T A\vec{x} = Q^T\vec{b}$$

or

$$R\vec{x} = Q^T\vec{b} .$$

We have

$$Q^T\vec{b} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \end{bmatrix} ,$$

and we solve using back substitution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{2}{9} + 3z \\ \frac{2}{9} - 2z \\ z \end{bmatrix} = \frac{2}{9}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Problem 9: True or False:

a) The column space does not change under row reduction. FALSE

b) A matrix with full column rank has a trivial null space. TRUE

c) A matrix that has a left inverse has full column rank. TRUE

d) A matrix that has full row rank is invertible. FALSE

e) Three vectors of which any two vectors are linearly independent are linearly independent. FALSE