PRACTICE TEST 2

Problem 1: Calculate the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 6 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix}$$

You do not have to calculate the eigenvectors. Is this matrix diagonalizable?

Problem 2: Show that any hermitean 2×2 matrix can be written in a unique way as

$$aI_2 + b\sigma_1 + c\sigma_2 + d\sigma_2$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; , \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; , \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the three Pauli matrices and $a, b, c, d \in \mathbb{R}$.

Problem 3: Let A be an $n \times n$ matrix. Compute

$$\frac{d}{dt}\det(I + tA)\Big|_{t=0} \; .$$

Problem 4: Solve the three term recursion, i.e., find a_n,

$$a_{n+1} = a_n + 2a_{n-1} \; , n = 0, 1, 2, \ldots$$

with the initial conditions $a_0 = a_1 = 1$.

Problem 5: Diagonalize the matrix

$$A = \begin{bmatrix} 2 & 4 - 3i \\ 4 + 3i & 2 \end{bmatrix}$$

by finding a unitary 2×2 matrix such that $A = UDU^*$ where D is diagonal.

Problem 6: Diagonalize the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

using orthogonal matrices, i.e., find D diagonal and R orthogonal so that $A = RDR^T$. (Hint: Guess one eigenvector.)

Problem 7: Compute the singular value decomposition for the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} .$$

Problem 8: Solve the differential equation

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \ , \ \vec{x}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \ , \ A = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}$$

Problem 10: True or false:
a) Every matrix is diagonalizable.
b) If λ is an eigenvalue of the $n \times n$ matrix A and μ an eigenvalue of the $n \times n$ matrix B then $\lambda + \mu$ is an eigenvalue of the matrix $A + B$. c) The eigenvectors of a symmetric matrix can be chosen to be orthogonal.
d) A three by three matrix has the eigenvalues $1, 2, 3$. Is it diagonalizable.
e) A symmetric four by four matrix has the eigenvalues 1 and 2. Is it diagonalizable?