SOLUTIONS OF PRACTICE TEST 2

Problem 1: Calculate the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 6 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix}$$

You do not have to calculate the eigenvectors. Is this matrix diagonalizable?

Solution: We have to compute the determinant of

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 3 & 4 \\ 0 & 2-\lambda & 5 & 6 \\ 0 & 0 & 3-\lambda & 4 \\ 0 & 0 & 4 & 3-\lambda \end{bmatrix}$$

Expanding according to the first column (remember the determinant of a matrix equals the determinant of its transposed) yields for the characteristic polynomial

$$(1-\lambda)(2-\lambda)\det \begin{bmatrix} 3-\lambda & 4 \\ 4 & 3-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda)[(3-\lambda)^2 - 16] \; .$$

The roots are easy:

$$(3-\lambda)^2 - 16 = 0$$

yields the roots 7, -1 which together with the other 1, 2 yields all the eigenvalues. The eigenvalues are all distinct and hence we have four linearly independent eigenvectors and hence the matrix is diagonalizable.

Problem 2: Show that any Hermitean 2×2 matrix can be written in a unique way as

$$aI_2 + b\sigma_1 + c\sigma_2 + d\sigma_3$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; , \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; , \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the three Pauli matrices and $a, b, c, d \in \mathbb{R}$.

Solution: The general Hermitean matrix is given by

$$\begin{bmatrix} \alpha & \gamma - i\delta \\ \gamma + i\delta & \beta \end{bmatrix}$$

where $\alpha, \beta, \gamma, \delta$ are real. We can write this as

$$\begin{bmatrix} \frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2} & \gamma - i\delta \\ \gamma + i\delta & \frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2} \end{bmatrix}$$

which equals

$$\frac{\alpha+\beta}{2} I_2 + \gamma\sigma_1 + \delta\sigma_2 + \frac{\alpha-\beta}{2}\sigma_3 \ .$$

We have to show that this representation is unique. This amounts to show that if

$$aI_2 + b\sigma_1 + c\sigma_2 + d\sigma_3 = 0$$

then $a = b = c = d = 0$. Clearly

$$aI_2 + b\sigma_1 + c\sigma_2 + d\sigma_3 = \begin{bmatrix} a+d & b-ic \\ b+ic & a-d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which implies the result. In other words the Pauli matrices together with the identity form a basis for the Hermitean matrices.

Problem 3: Let A be an $n \times n$ matrix. Compute

$$\frac{d}{dt}\det(I + tA)\Big|_{t=0} \ .$$

Solution: Write

$$\det(I + tA) = \sum_{\pi \in \mathcal{S}_n} \det P_\pi (\delta_{1\pi(1)} + tA_{1\pi(1)})(\delta_{2\pi(2)} + tA_{2\pi(2)}) \cdots (\delta_{n\pi(n)} + tA_{n\pi(n)})$$

where $\delta_{ij} = 0$ when $i \neq j$ and $\delta_{ii} = 1$. Differentiating with respect to t using the product rule we get upon setting $t = 0$

$$\frac{d}{dt}\det(I + tA)\Big|_{t=0} = \sum_{\pi \in \mathcal{S}_n} \sum_{k=1}^{n} \det P_\pi \delta_{1\pi(1)}\delta_{2\pi(2)} \cdots A_{k\pi(k)} \cdots \delta_{n\pi(n)}$$

$$= \sum_{k=1}^{n} \sum_{\pi \in \mathcal{S}_n} \det P_\pi \delta_{1\pi(1)}\delta_{2\pi(2)} \cdots A_{k\pi(k)} \cdots \delta_{n\pi(n)}$$

The element $\delta_{i,\pi(i)}$ is not equal to zero only if $\pi(i) = i$ and hence for

$$\delta_{1\pi(1)}\delta_{2\pi(2)} \cdots A_{k\pi(k)} \cdots \delta_{n\pi(n)}$$

not to be zero requires that π is the identity permutation. Hence the sum over all permutations collapses to a single term and we get the memorable formula

$$\frac{d}{dt}\det(I + tA)\Big|_{t=0} = \sum_{k=1}^{n} A_{kk} = \mathrm{Tr}A \ .$$

Problem 4: Solve the three term recursion, i.e., find a_n,

$$a_{n+1} = a_n + 2a_{n-1} \ , n = 0, 1, 2, \ldots$$

with the initial conditions $a_0 = a_1 = 1$.

Solution: We write

$$\vec{X}_n = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}$$

and get

$$\vec{X}_{n+1} = A\vec{X}_n \ , \vec{X}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \ .$$

Hence

$$\vec{X}_n = A^{n-1}\vec{X}_1 \ .$$

The eigenvalues are $2, -1$ and the corresponding eigenvectors

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Set

$$V = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

so that

$$AV = VD \text{ or } A = VDV^{-1}$$

where

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \ .$$

Hence

$$A^{n-1} = VD^{n-1}V^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2^{n-1} & 0 \\ 0 & (-1)^{n-1} \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2^n + (-1)^{n-1} & 2^n + 2(-1)^n \\ 2^{n-1} + (-1)^n & 2^{n-1} + 2(-1)^{n-1} \end{bmatrix}$$

and

$$A^{n-1}\vec{X}_1 = \frac{1}{3} \begin{bmatrix} 2^{n+1} + (-1)^n \\ 2^n + (-1)^{n-1} \end{bmatrix} \ ,$$

and $a_n = 2^{n+1} + (-1)^n$.

Problem 5: Diagonalize the matrix

$$A = \begin{bmatrix} 2 & 4-3i \\ 4+3i & 2 \end{bmatrix}$$

by finding a unitary 2×2 matrix such that $A = UDU^*$ where D is diagonal.

Solution: The matrix is Hermitean. Its characteristic polynomial is given by

$$\lambda^2 - 4\lambda + (4 - (4-3i)(4+3i)) = \lambda^2 - 4\lambda - 21 = (\lambda - 2)^2 - 25 = 0$$

so that the roots are given by

$$7 \ , \ -3 \ .$$

For the eigenvectors we solve $(A - \lambda I)\vec{v} = 0$. For the eigenvalue 7 we get the equation

$$-5a + (4-3i)b = 0\ ,\ (4+3i)a - 5b = 0$$

These two equations are equivalent (check!) and hence it suffices to consider the first on. If we set $a = (4-3i)$ and $b = 5$ we have a solution

$$\begin{bmatrix} (4-3i) \\ 5 \end{bmatrix}$$

Normalizing it yields the complex vector

$$\vec{w}_1 = \frac{1}{5\sqrt{5}} \begin{bmatrix} (4-3i) \\ 5 \end{bmatrix}$$

for the other eigenvalue -3 we have to solve the equation

$$5a + (4-3i)b = 0$$

which yields

$$\vec{w}_2 = \frac{1}{5\sqrt{5}} \begin{bmatrix} (4-3i) \\ -5 \end{bmatrix}$$

The inner product $\langle \vec{w}_1, \vec{w}_2 \rangle = 0$ (check!) The matrix

$$U = \frac{1}{5\sqrt{5}} \begin{bmatrix} (4-3i) & (4-3i) \\ 5 & -5 \end{bmatrix}$$

is unitary, i.e., $UU^* = U^*U = I$ (check!) and we have that

$$AU = U \begin{bmatrix} 7 & 0 \\ 0 & -3 \end{bmatrix}$$

or

$$A = U \begin{bmatrix} 7 & 0 \\ 0 & -3 \end{bmatrix} U^*$$

Problem 6: Diagonalize the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

using orthogonal matrices, i.e., find D diagonal and R orthogonal so that $A = RDR^T$. (Hint: Guess one eigenvector.)

Solution: The matrix is symmetric. The normalized eigenvector in question is

$$\vec{v}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and the corresponding eigenvalue is 6. Next we compute the characteristic polynomial

$$\det \begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & 3-\lambda & 1 \\ 3 & 1 & 2-\lambda \end{bmatrix}$$

$$= (1-\lambda)[(3-\lambda)(2-\lambda)-1] - 2[2(2-\lambda)-3] + 3[2-3(3-\lambda)]$$
$$= (1-\lambda)[5-5\lambda+\lambda^2] - 2[1-2\lambda] + 3[-7+3\lambda]$$
$$= 5-5\lambda+\lambda^2-5\lambda+5\lambda^2-\lambda^3-2+4\lambda-21+9\lambda$$
$$= -\lambda^3+6\lambda^2+3\lambda-18$$

Dividing by $(\lambda-6)$ yields

$$[-\lambda^3+6\lambda^2+3\lambda-18] : (\lambda-6) = -\lambda^2+3$$

and the eigenvalues are 6, $\sqrt{3}$ and $-\sqrt{3}$. To compute the eigenvector for $\sqrt{3}$ we row reduce

$$\begin{bmatrix} 1-\sqrt{3} & 2 & 3 \\ 2 & 3-\sqrt{3} & 1 \\ 3 & 1 & 2-\sqrt{3} \end{bmatrix}$$

to

$$\begin{bmatrix} -2 & 2(1+\sqrt{3}) & 3(1+\sqrt{3}) \\ 0 & 2 & 1+\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}$$

which yields the normalized eigenvector

$$\frac{1}{2\sqrt{3}} \begin{bmatrix} \sqrt{3}-1 \\ -\sqrt{3}-1 \\ 2 \end{bmatrix}$$

Repeating the computation for the eigenvalue $-\sqrt{3}$ yields

$$\frac{1}{2\sqrt{3}} \begin{bmatrix} -\sqrt{3}-1 \\ \sqrt{3}-1 \\ 2 \end{bmatrix}$$

Hence we have that

$$A = [\vec{v}_1, \vec{v}_2, \vec{v}_3] \begin{bmatrix} 6 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & -\sqrt{3} \end{bmatrix} \begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vec{v}_3^T \end{bmatrix}$$

Problem 7: Compute the singular value decomposition for the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} .$$

Solution: The matrix has rank 2. There are two possible ways to start. Either we diagonalize A^TA or AA^T, both yield the singular values. The second possibility is easier since the matrix is 2×2 and not 3×3.

$$AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The normalized eigenvectors are

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} , \ \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

These vectors $\vec{u}_1, \vec{u}_2$ are an orthonormal basis for the column space of A. Next we find and orthonormal basis for the column space for A^T by computing

$$\vec{v}_1 = \frac{1}{\sqrt{3}} A^T \vec{u}_1 = \frac{1}{\sqrt{3}\sqrt{2}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} , \ \vec{v}_2 = A^T \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The matrix

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}$$

and the SVD is given by $A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T$

$$A = \sqrt{3} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{3}\sqrt{2}} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$$

Problem 8: Solve the differential equation

$$\frac{d}{dt} \vec{x}(t) = A \vec{x}(t) \ , \ \vec{x}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \ , \ A = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}$$

Solution: The matrix A has the eigenvalues 0 and -5 and the corresponding eigenvectors are

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} , \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

There is no point in normalizing the vectors since the matrix A is not symmetric. The general solution is

$$\vec{x}(t) = a \begin{bmatrix} 3 \\ 2 \end{bmatrix} + b e^{-5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and we need to choose the numbers a, b to match the initial conditions

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix} = a \begin{bmatrix} 3 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

This can be easily solved and yields $a = b = 1$. Hence

$$\vec{x}(t) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + e^{-5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} .$$

Problem 9: True or false:
a) Every matrix is diagonalizable. FALSE
b) If λ is an eigenvalue of the $n \times n$ matrix A and μ an eigenvalue of the $n \times n$ matrix B then $\lambda + \mu$ is an eigenvalue of the matrix $A + B$. FALSE
c) The eigenvectors of a symmetric matrix can be chosen to be orthogonal. TRUE
d) A three by three matrix has the eigenvalues $1, 2, 3$. Is it diagonalizable. TRUE
e) A symmetric four by four matrix has the eigenvalues 1 and 2. Is it diagonalizable? YES