TEST 1, MATH 3406 N, SEPTEMBER 27, 2018

Name:

This test is to be taken without calculators and notes of any sort. The allowed time is 75 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write 1.414.... State your work clearly, otherwise credit cannot be given. Likewise, write legibly!

Problem 1:

a) (7 points) Row reduce the matrix below to reduced echelon form.

$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 5 & 4 \\ 0 & 0 & 3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 0 & -3 & -6 \\ 0 & 0 & 3 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b) (3 points) Circle the pivots in the final matrix. the ones in the first and third column.

c) (3 points) Write down the pivot columns of the original matrix.

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}$$

d) (2 points) Indicate the free variables. Second and fourth.

Problem 2:

a) (10 points) Using one step in the row reduction algorithm, find the LU factorization of the matrix

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 7 & 1 \end{bmatrix}$$

Multiply A by

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

so that

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 2 & 7 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

so that

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

b) (10 points) A matrix A has an LU factorization

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

Solve the system $A\vec{x} = \vec{b}$ where

$$\vec{b} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$$

Solve $L\vec{y} = \vec{b}$ using forward substitution. $x = -7$, $y = 5 + x = -2$, $z = 2 + 5y - 2x = 2 - 10 + 14 = 6$, so

$$\vec{y} = \begin{bmatrix} -7 \\ -2 \\ 6 \end{bmatrix}$$

Now solve $U\vec{x} = \vec{y}$ using back substitution. $z = -6$, $-2y = -2 + z$ or $y = 4$, $3x = -7 + 7y + 2z = 9$ or $x = 3$

Problem 3: Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 0 & 2 \\ -1 & 1 & -4 \end{bmatrix}.$$

Row reduction:

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

a) (4 points) Find a basis for $C(A)$.

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

b) (3 points) Find a basis for $N(A)$ Free variable is z so $x = -z, y = 3z$ and hence

$$\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

is a basis for the null space.

c) (3 points) Find a basis for $C(A^T)$ Choose two vectors perpendicular to the vector in the null space:

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

d) (5 points) Find a basis for $N(A^T)$. Must be perp to the two basis vectors in $C(A)$.

$$\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

Problem 4: a) (10 points) Use the normal equations to find the vector $\vec{x} \in \mathbb{R}^2$ such that $A\vec{x}$ is closest to $\vec{b}$, where

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ -1 & 1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

$$A^T A = \begin{bmatrix} 6 & -4 \\ -4 & 6 \end{bmatrix}, \quad A^T \vec{b} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{7}{5} \\ \frac{3}{5} \end{bmatrix}$$

b) (5 points) Find the projection of $\vec{b}$ onto the column space of A.

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \frac{2}{5} \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}$$

Problem 5: a) (10 points) Find the matrix for the orthogonal projection onto the space S spanned by the two orthonormal vectors

$$\vec{v}_1 = \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \text{ and } \vec{v}_2 = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

$$Q = \frac{1}{3}\begin{bmatrix} -2 & -1 \\ 1 & 2 \\ 2 & -2 \end{bmatrix}, Q^T = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ -1 & 2 & -2 \end{bmatrix}$$

$$QQ^T = \frac{1}{9}\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$$

b) (5 points) Find the matrix for the orthogonal projection onto $S^\perp$.

$$I - QQ^T = \frac{1}{9}\begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

c) (5 points) Find the least square approximations for $A\vec{x} = \vec{b}$, i.e., find all vectors $\vec{x} \in \mathbb{R}^3$ so that $A\vec{x}$ is closest to $\vec{b}$ where $\vec{b} = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$ and A is given by its QR factorization, i.e.,

$$A = \frac{1}{3}\begin{bmatrix} -2 & -1 \\ 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 3 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Solve $R\vec{x} = Q^T\vec{b}$

$$Q^T\vec{b} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

$-y + z = 0$, or $z = y$, $3x + y - z = -3$ or $x = -1$. So the solutions are

$$\begin{bmatrix} -1 \\ z \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Problem 6: (10 points) Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

the solution is

$$\vec{q}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{q}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{u}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Problem 7: True or False: (3 points each)

a) A matrix that has a zero null space always has a right inverse FALSE

b) If $A = QR$, Q orthogonal and R upper triangular then the column vectors of Q form an orthonormal basis for $C(A)$. TRUE

c) If the column vectors of A are linearly independent then the matrix A^TA is invertible. TRUE

d) The rank of a matrix A is the same as the rank of the matrix A^T. TRUE

e) For any two matrices A, B, if AB is invertible then both A and B are invertible. FALSE