TEST 2, MATH 3406 N, NOVEMBER 8, 2018

Name:

This test is to be taken without calculators and notes of any sort. The allowed time is 75 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. State your work clearly, otherwise credit cannot be given. Likewise, write legibly!

Problem 1: Compute the eigenvalues and eigenvectors of the following matrices and decide whether they are diagonalizable.

$$\text{a) } (\textbf{5 points}) \begin{bmatrix} 5 & 1 \\ -4 & 1 \end{bmatrix}$$

3 is a double eigenvalue and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is up to scaling the only eigenvector. Hence the matrix cannot be diagonalized.

$$\text{b) } (\textbf{10 points}) \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

The matrix is symmetric and hence can be diagonalized. 1 is an eigenvalue with eigenvector

$$\vec{v}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The simples way to continue it to write the matrix as

$$-2I_3 + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

We see that every vector orthogonal to $\vec{v}_1$ is an eigenvector with eigenvalue -2. E.g.,

$$\frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

are together with $\vec{v}_1$ from an orthonormal basis.

One can also compute the characteristic polynomial and get $-(\lambda+2)^2(\lambda-1)$ and then the rest is the same as before.

Problem 2: (10 points) Solve the following system of equations. Write the solutions in the form $z = a + ib, w = c + id$ where a, b, c, d are real.

$$\begin{aligned} (1+i)z + w &= 1 \\ z - (1-i)w &= 1 \end{aligned}$$

The solutions are $z = \frac{2-i}{3}$, $w = -\frac{i}{3}$.

Problem 3: (10 points) Compute the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{bmatrix} 4 & -3i \\ 3i & -4 \end{bmatrix}$$

The eigenvalues are 5 and -5. The corresponding eigenvectors

$$\frac{1}{\sqrt{10}} \begin{bmatrix} -3i \\ 1 \end{bmatrix} , \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3i \end{bmatrix}$$

(5 points) Find a unitary matrix U that diagonalizes A.

The matrix is

$$U = \frac{1}{\sqrt{10}} \begin{bmatrix} -3i & 1 \\ 1 & -3i \end{bmatrix}$$

Problem 4: (15 points) Find a_n given the three term recursion

$$a_{n+1} = 3a_n + 4a_{n-1} \ , \ n = 1, 2, \dots \ , \text{ and } a_0 = 1 \ , \ a_1 = 4 \ .$$

We write

$$\vec{X}_n = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}$$

Then

$$\vec{X}_{n+1} = A\vec{X}_n$$

where

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$$

Eigenvalues and eigenvectors are

$$4 \ , \ \begin{bmatrix} 4 \\ 1 \end{bmatrix} \ ; \ -1 \ , \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Note that the initial condition is $\vec{X}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and hence

$$\vec{X}_n = 4^{n-1} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

and hence $a_n = 4^n$.

Problem 5: (15 points) Compute the singular value decomposition for the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The easiest way is to compute AA^T which is

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

which has the eigenvectors

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ , \ \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

with the corresponding eigenvalues $\sigma_2^2 = 3$, $\sigma_2^2 = 1$. Now

$$\vec{v}_1 = \frac{1}{\sqrt{3}} A\vec{U}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \ , \ \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Hence we have

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{6}} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -\sqrt{3} & \sqrt{3} \end{bmatrix}$$

Check your answer!

Problem 6: a) **(10 points)** Find the general solution of the system of differential equations

$$\frac{dx}{dt} = x + 2y$$
$$\frac{dx}{dt} = 2x + y$$

Diagonalize the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

It has eigenvalues 3 and -1 with eigenvectors

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The general solution is

$$\vec{x}(t) = Ae^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + Be^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

b) **(5 points)** Find the solution that satisfies the initial condition $x(0) = 1$, $y(0) = 2$.
To satisfy the initial conditions we have to solve

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} + B\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

which yields $A = \frac{3}{2}$ and $B = -\frac{1}{2}$ The solution is

$$\vec{x}(t) = \frac{3}{2}e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Problem 7: True or false: **(3 points each)**

a) If a 2×2 matrix, that has a double eigenvalue, can be diagonalized then it must be proportional to I_2. TRUE

b) Every symmetric positive definite matrix $S = M^T M$ for some matrix M. TRUE

c) If A is any $m \times n$ matrix then $A^T A$ and AA^T have the same eigenvalues. FALSE

d) If A is an $n \times n$ matrix and $\vec{v} \in \mathbb{R}^n$ is any vector, then $\vec{v}, A\vec{v}, A^2\vec{v}, \ldots, A^n\vec{v}$ are linearly dependent. TRUE

e) If A B are $n \times n$ and A has an eigenvalue λ and B an eigenvalue μ, then AB has the eigenvalue $\lambda\mu$. FALSE