BASIS IN A FINITE DIMENSIONAL VECTOR SPACE

Hopefully you have come to appreciate the notion of a basis. In this note we prove the existence of a basis. We restrict ourselves to vector spaces V who are spanned by finitely many vectors

$$S = \{u_1, u_2, \ldots, u_k\} \ .$$

Recall that this means that every vector in V is a linear combination of the vectors $u_1, u_2, \ldots, u_k$. Also recall that a set of vectors is a basis for a vector space V if this set of vectors is linearly independent and spans the space V.

Theorem 0.1. *The vector space V has a basis of the form*

$$B = \{u_{i_1}, u_{i_2}, \ldots, u_{i_\ell}\} \ .$$

In particular B is a subset of S.

Proof. The reasoning here is inductive, not very difficult but important. We may assume that the set S does not contain the zero vector. If it does, just discard it. Next, we think of the vectors in S as an ordered list, there is a first vector, a second vector and so on until the kth vector. Now we pick u_1 as the first vector in our basis to be constructed. Consider the second vector u_2. If it is proportional to u_1 discard it from the list and go to the next one. If this vector is again proportional to u_1 discard it from the list and consider the next one. It may happen that you discard all the vectors following u_1 but then u_1 is a basis for the space V, since $u_1 \neq 0$ and every vector in V is proportional to u_1. If this does not happen then you end up with a vector in the list that is not proportional to u_1. We call it u_{i_2}. Now form the *span* T_2 of u_1 and u_{i_2}, i.e., the set of all linear combinations of u_1 and u_{i_2} and pick the next vector in the list. If this vector is in T_2 remove it from the list and check with the next vector as before. The first vector you find that is not in T_2 you call u_{i_3}. Now form the span T_3 of the vectors u_1, u_{i_2} and u_{i_3} and by discarding if necessary all the following vectors that are in T_3 pick the vector u_{i_4} that is not in T_3. The span of $u_1, u_{i_2}, u_{i_3}, u_{i_4}$ is T_4. One can go on with this process but it will stop once we have used up all the vectors. This provides you with a list of vectors $B = \{u_1, u_{i_2}, \ldots, u_{i_\ell}\}$. If we denote the space spanned by the vector u_1 by T_1 then we have a sequence of nested subspaces

$$T_1 \subset T_2 \subset T_3 \subset \cdots \subset T_\ell$$

and it never happens that $T_i = T_{i+1}$. Now we show that the set B consists of linearly independent vectors. Suppose that

$$\sum_{j=1}^{\ell} c_j u_{i_j} = 0 \ ,$$

that is

$$c_1 u_1 + c_2 u_{i_2} + \cdots + c_\ell u_{i_\ell} = 0 \ .$$

If $c_\ell \neq 0$ then u_{i_ℓ} is a linear combination of the other vectors and hence in $T_{\ell-1}$ contrary to our construction. Hence $c_\ell = 0$. For the same reason the next number $c_{\ell-1}$ must also be zero because otherwise $u_{\ell-1}$ would be in $T_{\ell-2}$. Continuing in this way we see that all coefficients have to vanish. Thus the vectors in B form a linearly independent set. By construction every vector in S is a linear combination of vectors in B and since the vectors in S form a spanning set, B is also a spanning set. Thus B is a basis. □

Theorem 0.2. *Assume that $\vec{u}_1, \dots, \vec{u}_n$ span a vector space V. Then any set of $n+1$ vectors in V is linearly dependent.*

Proof. We proceed by induction on n. For $n = 1$ every vector in V is proportional to $\vec{u}_1$ and hence any two vectors $\vec{f}_1, \vec{f}_2$ are of the form $f_1 = c_1\vec{u}_1$ and $\vec{f}_2 = c_2\vec{u}_1$ for numbers c_1, c_2. We find that $c_2\vec{f}_1 - c_1\vec{f}_2 = \vec{0}$. Thus, we assume as our induction hypotheses that the statement holds for any $k \le n-1$ and we have to prove it for $k = n$. Let $\vec{f}_1, \dots, \vec{f}_{n+1}$ vectors in V. We may write

$$\vec{f}_j = \sum_{i=1}^{n} c_{ij}\vec{u}_i \ , \ j = 1, \dots, n+1 \ ,$$

since $\vec{u}_1, \dots, \vec{u}_n$ span V. There exists at least one number among $c_{i(n+1)}, i = 1, \dots, n$ which is not zero, because otherwise $\vec{f}_{n+1}$ is the zero vector which implies that the set is linearly dependent and there is nothing to prove. Hence, we may assume that $\vec{f}_{n+1} \neq \vec{0}$ and we may assume after relabeling the vectors $\vec{u}_j, j = 1, \dots, n$ that $c_{n(n+1)} \neq 0$. We may solve for $\vec{u}_n$ and write

$$\vec{u}_n = \frac{1}{c_{n(n+1)}}\left[\vec{f}_{n+1} - \sum_{i=1}^{n-1} c_{i(n+1)}\vec{u}_i\right]$$

and hence

$$\vec{f}_j = \sum_{i=1}^{n-1} c_{ij}\vec{u}_i + \frac{c_{nj}}{c_{n(n+1)}}\left[\vec{f}_{n+1} - \sum_{i=1}^{n-1} c_{i(n+1)}\vec{u}_i\right] \ , \ j = 1, \dots, n \ ,$$

or

$$\vec{f}_j - \frac{c_{nj}}{c_{n(n+1)}}\vec{f}_{n+1} = \sum_{i=1}^{n-1}\left[c_{ij} - \frac{c_{nj}c_{i(n+1)}}{c_{n(n+1)}}\right]\vec{u}_i \ , \ j = 1, \dots, n \ ,$$

Thus, the n vectors $\vec{f}_j - \frac{c_{nj}}{c_{n(n+1)}}\vec{f}_{n+1}, j = 1, \dots, n$ are in the span of the vectors $\vec{u}_1, \dots, \vec{u}_{n-1}$ and hence by induction assumption they must be linearly dependent. This implies that the vectors $\vec{f}_1, \dots, \vec{f}_{n+1}$ are also linearly dependent. □

Corollary 0.3. *Let $\vec{u}_1, \dots, \vec{u}_k$ be a basis for the vector space V and $\vec{w}_1, \dots, \vec{w}_\ell$ be another basis for V. Then $k = \ell$ and we call this number the dimension of V,* $\dim V$.

Proof. If $\ell > k$, then the vectors $\vec{w}_1, \dots, \vec{w}_\ell$ are linearly dependent and hence is not from a basis contradicting the assumption. If $k > \ell$ the reasoning is similar. □

The following statement is often used.

Corollary 0.4. *If V, W are two vector spaces with $V \subset W$ and if* $\dim V = \dim W$*, then $V = W$.*

Proof. Otherwise we could find another vector in W which is not in V which can be added to the basis of V with the result that $\dim V < \dim W$. □