PRACTICE TEST 2

Problem 1: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that maps the vector $\vec{e}_1$ to the vector $\vec{e}_1 + \vec{e}_2$ and the vector $\vec{e}_2$ to $\vec{e}_1 - \vec{e}_2$.

a) Write the matrix associated with this transformation.

b) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection about the $x = y$ axis.

Write the matrix for the map $T \circ S$ as wll as the matrix associated with the map $S \circ T$. Sketch a rough image of what these transformations are doing to the standard basis vectors.

Problem 2: The eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 6 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix}$$

are:

(1) 1, 2, 7
(2) -1, 1, 2, 5
(3) 1, 2, 3
(4) -1, 1, 2, 7
(5) 1, 2, 3, -4

You do not have to calculate the eigenvectors.
Is this matrix diagonalizable?

Problem 3: Consider the parallelepiped formed by the three vectors

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ \vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

a) write its volume.

b) Suppose that the parallelepiped is sheared in the direction $\vec{d} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, i.e., the vectors $\vec{u}_1$ and $\vec{u}_2$ remain the same but the vector $\vec{u}_3$ is changed to $\vec{u}_3 + \vec{d}$. How does the volume change?

Extra credit: Can you use a different description of volume (not determinants) to justify that the volume should not change by shearing?

Problem 4 ** Extra credit: A matrix M is Hermitian if $M = M^*$, i.e., it is equal to its own conjugate transpose. Show that any Hermitian 2×2 matrix can be written in a unique way as

$$aI_2 + b\sigma_1 + c\sigma_2 + d\sigma_3$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the three Pauli matrices and $a, b, c, d \in \mathbb{R}$.

Problem 5: Give an example of a matrix that has the eigenvalues 0 and 1; both eigenvalues have algebraic multiplicity 2; the eigenvalue 0 has the geometric multiplicity 1 and the eigenvalue 1 has the geometric multiplicity 2.

Problem 6: i) Write the permutation below as a sequence of swaps. What is the sign of the permutation?

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix}$$

ii) Compute the determinant of the matrix

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 7: The numbers 6 and $\sqrt{3}$ are eigenvalues for the matrix below. What is its third eigenvalue?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Problem 8: True or false: (5 points each; if you say something is False, 2 points are reserved for providing a counterexample.)

a) If a 3×3 matrix has the eigenvalue 2 with geometric multiplicity 3 then the matrix is $2I_3$.

b) A three by three matrix has the eigenvalues $1, 2, 3$. Is it diagonalizable.

c) Consider the two matrices

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Can they be simultaneously diagonalized, i.e., do they have all their eigenvectors in common?

d) A two by two matrix has determinant 4 and trace 4. Is it necessarily diagonalizable?

e) The algebraic multiplicity may be smaller than the geometric multiplicity.

f) If a 3×3 matrix has the eigenvalue 2 with algebraic multiplicity 3 then the matrix is $2I_3$.