PRACTICE TEST 2

Problem 1: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that maps the vector $\vec{e}_1$ to the vector $\vec{e}_1 + \vec{e}_2$ and the vector $\vec{e}_2$ to $\vec{e}_1 - \vec{e}_2$.

a) Write the matrix associated with this transformation.

b) Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection about the $x = y$ axis.

Write the matrix for the map $T \circ S$ as well as the matrix associated with the map $S \circ T$. Sketch a rough image of what these transformations are doing to the standard basis vectors.

Solution: a)

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ , \ T(\vec{e}_2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and hence the matrix is given by

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

For b) there are essentially two avenues to pursue. One is to note that since $S(\vec{e}_1) = \vec{e}_2 \ , \ S(\vec{e}_2) = \vec{e}_1$

$$T \circ S(\vec{e}_1) = T(\vec{e}_2) \ , \ T \circ S(\vec{e}_2) = T(\vec{e}_1)$$

and hence the matrix associated with $T \circ S$ is

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Similarly for $S \circ T$ we have

$$S \circ T(\vec{e}_1) = S(\vec{e}_1 + \vec{e}_2) = S(\vec{e}_1) + S(\vec{e}_2) = \vec{e}_2 + \vec{e}_1$$

and

$$S \circ T(\vec{e}_2) = S(\vec{e}_1 - \vec{e}_2) = S(\vec{e}_1) - S(\vec{e}_2) = \vec{e}_2 - \vec{e}_1$$

and we get the matrix

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} .$$

Another way is to observe that the matrix associated with $T \circ S$ is the product of the matrix associated with T and S which is

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} .$$

Similarly

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} .$$

Problem 2: The eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 6 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix}$$

are:

(1) 1, 2, 7
(2) -1, 1, 2, 5
(3) 1, 2, 3
(4) -1, 1, 2, 7
(5) 1, 2, 3, -4

You do not have to calculate the eigenvectors.
Is this matrix diagonalizable?

Solution: The answer is $-1, 1, 2, 7$. The characteristic polynomial is

$$(1-\lambda)(2-\lambda)[(3-\lambda)^2 - 16] = (1-\lambda)(2-\lambda)[(\lambda^2 - 6\lambda - 7] = (1-\lambda)(2-\lambda)(\lambda-7)(\lambda+1)\ .$$

Problem 3: Consider the parallelepiped formed by the three vectors

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\ ,\ \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\ ,\ \vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

a) write its volume.

b) Suppose that the parallelepiped is sheared in the direction $\vec{d} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, i.e., the vectors $\vec{u}_1$ and $\vec{u}_2$ remain the same but the vector $\vec{u}_3$ is changed to $\vec{u}_3 + \vec{d}$. How does the volume change?

Extra credit: Can you use a different description of volume (not determinants) to justify that the volume should not change by shearing?

Solutions: We have to compute the determinant of the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

which equals -1. Hence the volume is 1. For the part b) there are two ways to do it. We have to compute the determinant of the matrix $[\vec{u}_1, \vec{u}_2, \vec{u}_3 + \vec{d}]$. This becomes easy if one observes that the vector $\vec{d} = \vec{u}_1 + \vec{u}_2$ and hence using the linearity of the determinant in the columns yields

$$\det[\vec{u}_1, \vec{u}_2, \vec{u}_3 + \vec{u}_1 + \vec{u}_2] = \det[\vec{u}_1, \vec{u}_2, \vec{u}_3] + \det[\vec{u}_1, \vec{u}_2, \vec{u}_1] + \det[\vec{u}_1, \vec{u}_2, \vec{u}_2]$$

and the last two determinants are zero. Hence the volume does not change. The geometric explanation is that the shear does not change the height of the parallelepiped with base $\vec{u}_1, \vec{u}_2$.

Problem 4 ** Extra credit: A matrix M is Hermitian if $M = M^*$, i.e., it is equal to its own conjugate transpose. Show that any Hermitian 2×2 matrix can be written in a unique way as

$$aI_2 + b\sigma_1 + c\sigma_2 + d\sigma_3$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the three Pauli matrices and $a, b, c, d \in \mathbb{R}$.
Solution: Any hermitean 2×2 matrix can be written as

$$\begin{bmatrix} a & b - ic \\ b + ic & d \end{bmatrix}$$

which we can write as

$$\frac{a+d}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{a-d}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} .$$

These 4 matrices are obviously linearly independent because if we write the zero matrix in the above form we must have that $a = d = b = c = 0$. Hence the Pauli matrices together with the identity matrix form a basis for the linear space of hermitean 2×2 matrices.

Problem 5: Give an example of a matrix that has the eigenvalues 0 and 1; both eigenvalues have algebraic multiplicity 2; the eigenvalue 0 has the geometric multiplicity 1 and the eigenvalue 1 has the geometric multiplicity 2.
Solution: The eigenvalues 0 and 1 must have algebraic multiplicity 2 and hence we start with

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

However, the eigenvalue 0 in the above matrix has geometric multiplicity 2, because there are two linearly independent eigenvectors. Thus we try

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

from which we see that the only eigenvector (up to scaling) for the eigenvalue 0 is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

Problem 6: i) Write the permutation below as a sequence of swaps. What is the sign of the permutation?

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix}$$

ii) Compute the determinant of the matrix

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution: This one is a bit tricky: We start first with $1 \leftrightarrow 3$, i.e., 1 goes to 3 and 3 goes to 1, or

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 & 4 & 5 \end{pmatrix}$$

Next since we want that $2 \to 1$ we try $2 \leftrightarrow 1$ or

$$\pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 & 4 & 5 \end{pmatrix}$$

These two permutations together send $1 \to 3, 2 \to 1, 3 \to 2, 4 \to 4$ and $5 \to 5$, or

$$\pi_2 \circ \pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 & 4 & 5 \end{pmatrix}$$

Hence we have to make another swap, namely $2 \leftrightarrow 4$ or

$$\pi_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix}$$

Composing these three permutations $\pi_3 \circ \pi_2 \circ \pi_1$ yields the given one. This is therefore an odd permutation.

To solve ii), note first that this the determinant is the same as the one of the matrix

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This is a permutation matrix where the first column of the identity matrix goes to third place, the second column goes to first place, the third column goes to fourth place and fifth column is fixed. The determinant of this matrix is equals the sign of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix}$$

which we determined to be odd. Hence the determinant is -1.

Problem 7: The numbers 6 and $\sqrt{3}$ are eigenvalues for the matrix below. What is its third eigenvalue?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Solution: The trace of the matrix is the sum of the eigenvalues. The trace is 6 and since two of the eigenvalues are 6 and $\sqrt{3}$, the third one must be $-\sqrt{3}$.

Problem 8: True or false: (5 points each; if you say something is False, 2 points are reserved for providing a counterexample.)

a) If a 3×3 matrix has the eigenvalue 2 with geometric multiplicity 3 then the matrix is $2I_3$.
TRUE

b) A three by three matrix has the eigenvalues $1, 2, 3$. Is it diagonalizable.
TRUE

c) Consider the two matrices

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Can they be simultaneously diagonalized, i.e., do they have all their eigenvectors in common?
NO because they do not commute, i.e., $AB \neq AB$. We have learned in class that two matrices A, B can be simultaneously diagonalized if and only if $AB = BA$.

d) A two by two matrix has determinant 4 and trace 4. Is it necessarily diagonalizable?
NO. The following matrix is a counterexample:

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

e) The algebraic multiplicity may be smaller than the geometric multiplicity.
The algebraic multiplicity is always greater or equal the geometric multiplicity.

f) If a 3×3 matrix has the eigenvalue 2 with algebraic multiplicity 3 then the matrix is $2I_3$.
FALSE
The matrix below is a counterexample:

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$