TEST 1, MATH 3406 A, SEPTEMBER 26, 2019

Print Name:

Section Number:

This test is to be taken without calculators and notes of any sort. The allowed time is 75 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. State your work clearly, otherwise credit cannot be given. Likewise, write legibly!

I abide by the Georgia Tech honor code. Signature:

Problem 1:

a) (7 points) Row reduce the matrix below to reduced echelon form.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 3 & 1 & -1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -5 & -10 & -15 \\ 0 & -5 & -10 & -15 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b) (3 points) Circle the pivots in the final matrix.

c) (3 points) Write down the pivot columns of the original matrix.

$$\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

d) (2 points) Indicate the free variables. Third and fourth column.

Problem 2:

a) (5 points) Using one step in the row reduction algorithm, find the LU factorization of the matrix

$$A = \begin{bmatrix} 1 & 4 & 6 \\ 2 & 6 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 6 \\ 2 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 6 \\ 0 & -2 & -4 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 4 & 6 \\ 2 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 6 \\ 0 & -2 & -4 \end{bmatrix}$$

b) (10 points) A matrix A has an LU factorization

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solve the system $A\vec{x} = \vec{b}$ where

$$\vec{b} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$L\vec{y} = \vec{b}$

$$\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$U\vec{x} = \vec{y}$

$$\vec{x} = \begin{bmatrix} 3/2 \\ 0 \\ 1 \end{bmatrix}$$

Problem 3: Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 0 & 2 \\ 8 & 2 & 2 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

a) (4 points) Find a basis for $C(A)$.

$$\begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

b) (3 points) Find a basis for $N(A)$

$$\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

c) (3 points) Find a basis for $C(A^T)$

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

d) (5 points) Find a basis for $N(A^T)$

$$\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

Problem 4: a) (10 points) Use the normal equations to find the vector $\vec{x} \in \mathbb{R}^2$ such that $A\vec{x}$ is closest to $\vec{b}$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & 2 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

$$\vec{x}^* = \begin{bmatrix} 5/3 \\ -1/6 \end{bmatrix}$$

b) (5 points) Find the projection of $\vec{b}$ onto the column space of A.

$$\vec{b}^* = \frac{1}{2} \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$$

Problem 5: a) (10 points) Find the matrix for the orthogonal projection onto the space S spanned by the two orthonormal vectors

$$\vec{v}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \text{ and } \vec{v}_2 = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

$$QQ^T = \frac{1}{9}\begin{bmatrix} 5 & 2 & 4 \\ 2 & 8 & -2 \\ 4 & -2 & 5 \end{bmatrix}$$

b) (5 points) Find the matrix for the orthogonal projection onto $S^\perp$.

$$I - QQ^T = \frac{1}{9}\begin{bmatrix} 4 & -2 & -4 \\ -2 & 1 & 2 \\ -4 & 2 & 4 \end{bmatrix}$$

c) (5 points) Find the least square approximations for $A\vec{x} = \vec{b}$, i.e., find all vectors $\vec{x} \in \mathbb{R}^3$ so that $A\vec{x}$ is closest to $\vec{b}$ where $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and A is given by its QR factorization, i.e.,

$$A = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 2 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

$$Q^T\vec{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$R\vec{x} = Q^T\vec{b}$

$$\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 2/3 \\ -1 \\ 1 \end{bmatrix}$$

Problem 6: (10 points) Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \ \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \ \vec{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{v}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{q}_3 = \vec{u}_3 - \vec{v}_1 \cdot \vec{u}_3\vec{v}_1 - \vec{v}_2 \cdot \vec{u}_3\vec{v}_2 = \vec{u}_3 - \vec{v}_1 - \vec{v}_2$$

$$\vec{v}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Problem 7: True or False: (3 points each, no partial credit)

a) If the row vectors of A are linearly independent then the matrix AA^T is invertible. TRUE

The reason is that A^T is a matrix whose column vectors are linearly independent and hence $(A^T)^T A^T = AA^T$ is invertible.

b) If $A = QR$, Q orthogonal and R upper triangular then the column vectors of Q form an orthonormal basis for $C(A)$. TRUE

c) A matrix that has full column rank, i.e., every column has a pivot, always has a right inverse. FALSE

Again, take the matrix

$$A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

A right inverse B must be of the form

$$B = \begin{bmatrix} a & b \end{bmatrix}$$

hence

$$AB = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix},$$

which cannot be the two by two identity no matter what a, b.

d) The rank of a matrix A^T is the same as the rank of the matrix A. TRUE

e) For any two matrices A, B, if neither A nor B is invertible, then AB is not invertible either. FALSE

Take

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

so that $AB = 1$ which is certainly invertible.