TEST 1, MATH 3406 K, FEBRUARY 13, 2020

Print Name:

This test is to be taken without calculators and notes of any sort. The allowed time is 75 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. State your work clearly, otherwise credit cannot be given. Likewise, write legibly!

I abide by the Georgia Tech honor code. Signature:

Problem 1:

a) (7 points) Row reduce the matrix below to **reduced echelon** form.

$$A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 3 & 7 & 10 & 5 \\ 2 & 5 & 7 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & -4 \\ 0 & 1 & 1 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 11 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b) (3 points) Circle the pivots in the matrix you obtain in the final step of row reduction above.

c) (3 points) Write down a basis for the column space of A.

$$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 5 \end{bmatrix}$$

d) (5 points) Find a basis for $Nul(A)$.

$$\begin{bmatrix} -y - 11z \\ -y + 4z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -11 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

e) (2 points) What is the dimension of $Nul(A^T)$?

The rank of the matrix is 2, the row space has three rows and hence the dimension is 1.

Problem 2:

a) (10 points) A matrix A has an LU factorization

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & -8 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Solve the system $A\vec{x} = \vec{b}$ where

$$\vec{b} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$$

First solve

$$L\vec{y} = \vec{b}\ ,\ \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$$

$$U\vec{x} = \vec{y}$$

$$z = 1\ , y = 2\ , x = 2/3\ , \begin{bmatrix} 2/3 \\ 2 \\ 1 \end{bmatrix}$$

b) (3 points) What are the pivots in in the row reduced A?

$$3, -2, 2$$

c) (4 points) Write down L and U for the LU decomposition of A^T.
Write

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4/3 & -8/3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 4/3 & 1 & 0 \\ -8/3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 4/3 & 1 & 0 \\ -8/3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & 6 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

d) (extra credit: 3 points) What are the pivots of A^T?

$$3, -2, 2$$

e) (3 points) Give an example of a matrix that cannot be put into LU form.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Problem 3: a) (8 points) Use the normal equations to find the least squares solution to $Ax = b$. In other words, find vector $\vec{x} \in \mathbb{R}^2$ such that $A\vec{x}$ is closest to $\vec{b}$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}, \quad A^T\vec{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

2x+2y = 3 2x+3y=5

$$\vec{x} = \begin{bmatrix} -1/2 \\ 2 \end{bmatrix}.$$

b) (2 points) Find the projection of $\vec{b}$ onto the column space of A.

$$\vec{b}^* = A\vec{x} = \begin{bmatrix} 3/2 \\ 3/2 \\ 2 \end{bmatrix}$$

c) (5 points) Give an example of a matrix A and a vector b such that projection of b to Col(A) is always 0.

Take the vector

$$\vec{b} - \vec{b}^* = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

which is perpendicular to $C(A)$ since $A^T(\vec{b} - \vec{b}^*) = 0$.

Problem 4: a) (10 points) Find the matrix for the orthogonal projection onto the space S spanned by the two orthonormal vectors

$$\vec{v}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \text{ and } \vec{v}_2 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$Q = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 2 & -2 \end{bmatrix}$$

$$Q^T = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \end{bmatrix}$$

$$P_S = QQ^T = \frac{1}{9}\begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

b) (5 points) Find the matrix for the orthogonal projection onto $S^\perp$.

$$P_{S^\perp} = I - QQ^T = \frac{1}{9}\begin{bmatrix} 4 & -4 & -2 \\ -4 & 4 & 2 \\ -2 & 2 & 1 \end{bmatrix}$$

c) (5 points) Given a vector $\vec{u} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$, write down its orthogonal decomposition into components along S and $S^\perp$.

$$P_S\vec{u} = \frac{1}{9}\begin{bmatrix} 5p + 4q + 2r \\ 4p + 5q - 2r \\ 2p - 2q + 8r \end{bmatrix}$$

$$P_{S^\perp}\vec{u} = \frac{1}{9}\begin{bmatrix} 4p - 4q - 2r \\ -4p + 4q + 2r \\ -2p + 2q + r \end{bmatrix}$$

Problem 5: (10 points) Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \ , \ \vec{u}_2 = \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \end{bmatrix} \ , \ \vec{u}_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\vec{q}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} , \vec{a}_2 = \vec{u}_2 - (\vec{u}_2 \cdot \vec{q}_1)\vec{q}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix} , \vec{q}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix}$$

$$\vec{a}_3 = \vec{u}_3 - \vec{u}_3 \cdot \vec{q}_1\vec{q}_1 - \vec{u}_3 \cdot \vec{q}_2\vec{q}_2 = \vec{u}_3 - 2\vec{q}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} , \vec{q}_3 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Problem 6: True or False: (3 points each, no partial credit.)

a) If the row vectors of A are linearly independent then the matrix AA^T is invertible. TRUE

b) If a matrix has linearly independent rows, then its columns are also linearly independent. FALSE

c) A matrix that has full row rank, i.e., every row has a pivot, always has a right inverse. TRUE

d) If S is a k-dimensional subspace in $\mathbb{R}^n$, where $k < n$, then there can be vectors in S that form an orthonormal system of size n. FALSE

e) If $A^T A = 0$ matrix, then A must be the 0 matrix. TRUE

Problem 7: (Extra credit)

U is a subspace of $\mathbb{R}^n$. Let A be a matrix whose columns are vectors in U. Let B be a matrix whose rows are vectors in $U^\perp$.
a) (4 points) What can you say about BA?
It is the zero matrix

b) (3 points) Is the matrix product AB in general defined? No, not in general.