HOMEWORK 1 , SOLUTIONS OF SELECT PROBLEMS

Problem 1, (5 points): The symmetric difference of two sets A, B is defined to be

$$A \Delta B := (A \setminus B) \cup (B \setminus A) .$$

Show that

$$A \Delta B = (A \cup B) \setminus (A \cap B) .$$

Problem 2, (5 points): Let $f : X \to Y$ be a function between two sets X, Y. For any two sets $A, B \subset Y$ show that

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \ , \ f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

Problem 3, (5 points): With the same assumptions as in the previous problem, is it true that for any two sets $A, B \subset X$, $f(A \cup B) = f(A) \cup f(B)$? Is it true that $f(A \cap B) = f(A) \cap f(B)$?

Problem 4, (7 points): Recall that a metric space X is compact if and only if every open cover of X has a finite sub-cover. Prove that any sequence in a compact metric space has a convergent sub-sequence.

First step: Find a candidate for the limit. Let's start with an arbitrary sequence $x_n, n = 1, 2, 3, \dots$. If the *set* $S = \{x_1, x_2, x_3, \dots\}$ is finite (what is the difference between the set and the sequence?) then the sequence will, after finitely many terms be constant and hence converges. This was the easy case. Now to the more difficult one where S is not finite. Here we have to use compactness. I claim that there exists a point $x \in X$ so that every open ball with x as center contains infinitely many points of the sequence. To see this, assume the contrary, i.e., every point in $y \in X$ is the center of an open ball $B(y)$ that contains only finitely many points of the sequence. The union $\cup_{y \in X} B(y)$ is an open cover of the space X. This space is compact and hence it contains a finite sub-cover, i.e., $X = \cup_{j=1}^{N} B_j$. Each ball contains only finitely many points of the set S and hence the set S must be finite which is a contradiction. Thus, there exists a point x so that every ball centered at x, $B_\varepsilon(x)$ (the open ball of radius ε centered at x), contains infinitely many points of the set S. *It is this point x that is our candidate for the limit.*

Second step: Show that x is indeed the limit for a subsequence. Since every ball centered at x contains infinitely many points of the sequence we may choose n_1 so that $x_{n_1} \in B_1(x)$, then n_2 so that $x_{n_2} \in B_{1/2}(x)$ and so on so that $x_{n_k} \in B_{1/k}(x)$ for all $k = 1, 2, 3, \dots$. Hence, the sequence x_{n_k} converges to x as $k \to \infty$ and we are done.

Problem 5, (3 points): Prove that every compact metric space is complete.

Solution We have to show that any Cauchy sequence in X converges. Since X is a compact metric space, we know from Problem 3 that every sequence has a convergent subsequences. From Problem 1 we know that any Cauchy sequence, which has a convergent subsequence, converges.