HOMEWORK 2, SOLUTIONS

Problem 1, (5 points): Consider the set X of rational numbers in the interval $[0, 1]$. For any of the intervals $(a, b), [a, b], (a, b], [a, b) \subset [0, 1] \cap \mathbb{Q}$ define its measure to be

$$m(a, b) = b - a \ .$$

Show that this measure cannot be extended to a σ additive measure on this set.

Solution: The measure of the set $[0, 1] \cap \mathbb{Q}$ is 1. Suppose the measure has an extension to a sigma-additive measure. Each point in the set $[0, 1] \cap \mathbb{Q}$ has zero measure and since the set $[0, 1] \cap \mathbb{Q}$ is countable its measure is zero which is a contradiction.

Problem 2, (5 points): Recall that the symmetric difference of two sets $A, B \subset \mathbb{R}^d$ is given by $A\Delta B = (A \setminus B) \cup (B \setminus A)$. Prove that

$$||A|_e - |B|_e| \leq |A\Delta B| \ .$$

Solution: We have that $A \subset B \cup (A\Delta B)$ and hence by subadditivity $|A|_e \leq |B|_e + |A\Delta B|_e$ and hence $|A|_e - |B|_e \leq |A\Delta B|_e$. Interchanging A and B yields $|B|_e - |A|_e \leq |A\Delta B|_e$.

Problem 3, (5 points): Let $\{E_k\}_{k=1}^\infty$be a sequence of sets in $\mathbb{R}^d$. Define the sets

$$\limsup_{k\to\infty} E_k := \cap_{j=1}^\infty \left(\cup_{k=j}^\infty E_k\right) \ \text{and} \ \liminf_{k\to\infty} E_k := \cup_{j=1}^\infty \left(\cap_{k=j}^\infty E_k\right)$$

Show that $\limsup_{k\to\infty} E_k$ consists of those points $x \in \mathbb{R}^d$ that belong to infinitely many of the E_k and that $\liminf_{k\to\infty} E_k$ consists of the points $x \in \mathbb{R}^d$ that belong to all but finitely many of the E_k.

Solution: a) Let $x \in \cap_{j=1}^\infty \left(\cup_{k=j}^\infty E_k\right)$. This means that $x \in \cup_{k=j}^\infty E_k$ for all $j = 1, 2, \dots$. Thus, for every natural number j there exists a natural number $k \geq j$ such that $x \in E_k$. This means that x cannot belong only to finitely many of the sets E_k. Conversely, if x belongs to infinitely many sets E_k then for any j there exists $k \geq j$ so that $x \in E_k$. Hence, for every j, $x \in \cup_{k=j}^\infty E_k$ and hence $x \in \cap_{j=1}^\infty \left(\cup_{k=j}^\infty E_k\right)$.

b) If $x \in \cup_{j=1}^\infty \left(\cap_{k=j}^\infty E_k\right)$ then there exists some j so that $x \in \cap_{k=j}^\infty E_k$. Thus, there exists some j so that $x \in E_k$ for all $k \geq j$,i.e., $x \in E_k$ in all but finitely many k. Conversely, if $x \in E_k$ for all but finitely many k, then there exists j so that $x \in E_k$ for all $k \geq j$, i.e., there exists j, so that $x \in \cap_{k=j}^\infty E_k$ and hence $x \in \cup_{j=1}^\infty \left(\cap_{k=j}^\infty E_k\right)$.

Problem 4, (5 points): Please do problem 2.1.39 in Heil's book.

Solution: Suppose there exist countable many boxes with $\sum_k vol(Q_k) < \infty$ and each point of E belongs to infinitely many boxes. This means that

$$E \subset \cap_{j=1}^{\infty}(\cup_{k=j}^{\infty} Q_k)$$

and hence

$$|E|_e \leq |\cup_{k=j}^{\infty} Q_k|_e$$

for all $j \geq 1$. Because, $\sum_{k=1}^{\infty} vol(Q_k) < \infty$

$$|E|_e \leq \sum_{k=j}^{\infty} vol(Q_k) \to 0$$

as $j \to \infty$. Conversely, assume that $|E_e = 0$. For any n there exist countably many boxes Q_n^k so that $E \subset \cup_{k=1}^{\infty} Q_n^k$ and such that

$$\sum_{k=1}^{\infty} vol(Q_n^k) < 2^{-n} \ .$$

Pick any $x \in E$, then $x \in \cup_{k=1}^{\infty} Q_n^k$ for all n and hence for any n there must be a $k(n)$ such that $x \in Q_n^k$. It could be that all but finitely many of these boxes are the same. However that cannot be, because if the were infinite repetitions of the same box the sum of the volumes could not be finite. Hence x belongs to infinitely may boxes.

Problem 5, (5 points): Please do problem 2.1.40 in Heil's book.

Solution: Assume the contrary, that for every real h, $Z + h$ contains a real number r. This means that for every h real there exists a rational number r such that

$$h \in r - Z \ .$$

Thus, every real number $h \in \cup_r(r - Z)$ where the union is over all rational numbers. Since the number of rational numbers is countable and since $|r - Z| = |Z| = 0$ we have that $|\cup_r (r - Z)| = 0$ and hence the measure of the set of real numbers is zero which is certainly not the case.