HOMEWORK 8 , DUE THURSDAY MARCH 12

Problem 1, (5 points): Please do Problem 4.5.24 in Heil
Solution: Note first that

$$n \ln(1 + \frac{f(x)}{n}) \le f(x) \ .$$

One way this can be seen is to note that for $a \ge 0$

$$\ln(1+a) = \int_1^{1+a} \frac{1}{t} dt \le \int_1^{1+a} dt = a \ .$$

Now

$$n \ln(1 + \frac{f(x)}{n}) \to f(x)$$

pointwise and $f(x)$ is integrable. The statement follows by dominated convergence.

Problem 2, (5 points): Please do Problem 4.5.27 in Heil
Solution: We trace everything back to Fatou's lemma. Assume that the f_ns are real and non-negative

$$\liminf \int g_n - f_n \ge \int g - f \ , \liminf \int f_n \ge \int f$$

from which we conclude, since $\int g_n \to \int g$ that $\lim \int f_n = \int f$. Suppose that the sequence f_n is real but not positive. Since $f_n^+, f_n^- \le g_n$ and since $f_n^\pm \to f^\pm$ we find by the previous argument that

$$\lim \int f_n^\pm = \int f^\pm$$

and hence $\lim \int f_n = \lim \int f_n^+ - f_n^- = \int f$ and $\lim \int |f_n| = \lim \int f_n^+ + f_n^- = \int |f|$. Suppose that f_n is complex. Then $f_n = a_n + ib_n$. We have that

$$|a_n|, |b_n| \le g_n$$

and a_n, b_n converge pointwise to a, b respectively where $f = a + ib$. The statements follow from the above.

Problem 3, (5 points): Please work Problem 4.6.12 in Heil
Solution: The contribution to the repeated integrals cancel and hence the repeated integrals vanish. The absolute value, however has the value $1/|Q_n|$ in every square Q_n and hence its integral over the square Q_n is 1. By countable subadditivity the integral over the whole square diverges. Hence the Lebesgue integral of the positive part of f and the negative part of f diverge and since the Lebesgue integral of f is the difference, the integral is not defined.

Problem 4, (5 points): Please work Problem 4.6.13 in Heil
Solution: $I_1 = 0$ and the integral $\int_{-1}^1 \frac{1}{1-y^2} dy = +\infty$.

Problem 5, (5 points): Please do Problem 4.6.19 in Heil

Solution:

The function under consideration is

$$F(x,t) = e^{-tx} \sin x \chi_{[0,a]}(x) \ .$$

Now

$$|F(x,t)| \leq e^{-tx} \chi_{[0,a]}(x)$$

which is integrable on $\mathbb{R}_+ \times \mathbb{R}_+$. Now,

$$\int_0^a e^{-tx} \sin x dx = \frac{1 - e^{-at}\cos a - te^{-ta}\sin a}{1+t^2}$$

and hence, by Fubini

$$\int_0^a \frac{\sin(x)}{x} dx = \int_0^\infty \frac{1 - e^{-at}\cos a - te^{-ta}\sin a}{1+t^2} dt$$

Moreover, we have pointwise

$$\lim_{a\to\infty} \frac{1 - e^{-at}\cos a - te^{-ta}\sin a}{1+t^2} = \frac{1}{1+t^2}$$

Further,

$$\Big|\frac{1 - e^{-at}\cos a - te^{-ta}\sin a}{1+t^2}\Big| \leq \frac{1+(1+t)e^{-at}}{1+t^2} \leq \frac{1+(1+t)e^{-t}}{1+t^2}$$

for $a > 1$. This function is integrable and hence

$$\lim_{a\to\infty} \int_0^\infty \frac{1 - e^{-at}\cos a - te^{-ta}\sin a}{1+t^2} dt = \int_0^\infty \frac{1}{1+t^2} dt = \frac{\pi}{2} \ ,$$

or

$$\lim_{a\to\infty} \int_0^a \frac{\sin x}{x} dx = \frac{\pi}{2}$$