PRACTICE MIDTERM EXAM

1. Topics

Some simple facts from standard real analysis, exterior measure, Lebesgue measure, countable additivity, Caratheodory's criterion, non-measurable sets, measurable functions, Egorov's and Luzin's theorem, the Lebesgue intergal for non-negative functions. Monotone convergence and Fatou's lemma. Thus the test might cover everything up to and including section 4.2.2 in Heil.

2. Help for the test

You may prepare a sheet both sides with information and bring it to the exam. Otherwise no help is allowed.

3. Practice Test

Problem 1: Let $C \subset \mathbb{R}^d$ be compact and $f : C \to \mathbb{R}$ an upper semicontinuous function. Prove that f attains its maximum.

Problem 2: Recall that

$$\liminf_{k\to\infty} E_k = \cap_{k=1}^{\infty}(\cup_{j=k}^{\infty} E_k)$$

Suppose that $\sum_{k=1}^{\infty} |E_k|_e < \infty$. Show that $\liminf_{k\to\infty} E_k$ has measure zero.

Problem 3: Let $E_j \subset \mathbb{R}^d, j = 1, 2, \ldots$ be a sequence of sets, not necessarily measurable. Assume that $E_j \subset E_{j+1}$ for $j = 1, 2, \ldots$. Prove that

$$| \cup_{j=1}^{\infty} E_j|_e = \lim_{j\to\infty} |E_j|_e \ .$$

Problem 4: Define the inner Lebesgue measure of a set $A \subset \mathbb{R}^d$ to be

$$|A|_i = \sup\{|F| : F \; rmis \; closed \; F \subset A\}$$

Prove that if A is Lebesgue measurable then $|A|_e = |A|_i$. Moreover, show that if $|A|_e < \infty$ and $|A|_e = |A|_i$, then A is Lebesgue measuarable.

Problem 5, (5 points): In Egorov's theorem we had to assume that $|E| < \infty$. Give an example of a sequence of functions on the whole real line which converges but where Egorov's theorem fails.

Problem 6, (5 points): Prove that $f : E \to [-\infty, \infty]$ is measurable if and only if

$$\{f > r\}$$

is measurable for every r rational.

Problem 7, (5 points): Assume Fatou's lemma and deduce the monotone convergence theorem from it.