Errata for the Second Edition of "Analysis", complete as of November 18, 2016

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Most of these errata refer to the first edition and the first two printings of the second edition. There is a third printing that appeared in August 2014, in which a few errata remain.

We thank the many friends and colleagues who took the trouble to tell us about errors and misprints.

Page 11. In the hypothesis of Theorem 1.4 it is necessary to require that the set $\mathcal{A}$ contains the empty set $\emptyset$ and the whole set Ω.

Page 13. In paragraph 6, where essential support is defined, the symbol Ω is incorrectly used. This symbol denotes the underlying measure space throughout section 1.5, except here, where it denotes a *collection* of open sets. To solve this problem change Ω to $\widetilde{\Omega}$ in this paragraph (only).

Page 16. Change Levi to Beppo Levi in the penultimate paragraph.

Page 17. Replace the second line by "for every x in a set $\Omega \sim \Theta$, where $\Theta \in \Sigma$ and $\mu(\Theta) = 0$. Let us redefine the f^j by setting $f^j(x) = 0$, for $x \in \Theta$, while $f_j(x)$ is unchanged for $x \notin \Theta$. This redefinition makes the sequence monotone for all $x \in \Omega$, but it does not change the integral $I_j = \int f^j d\mu$. For every x we can now define"

Delete the fourth line and the first part of the fifth line up to the period. On the sixth line replace "well defined a.e." by "well defined for all x. If $\mu(\{x : f(x) = \infty\}) > 0$ we say that f is not summable for the purpose of the following Theorem 1.6. In any case we set $f(x) = 0$ on the set $\{x : f(x) = \infty\}$ so that our new f is a function from $\Omega \to \mathbb{R}$, in conformity with the definition on the first line of 1.5."

Add a new first line to the proof: In the following, f^j and f refer to the sequence and the limit as redefined above.

Page 26. In Theorem 1.13 one has to assume that the measure space (Ω, Σ, μ) is sigma finite. Only then can one apply Fubini's theorem.

Page 28. In Theorem 1.14 (Bathtub principle) the space (Ω, Σ, μ) has to be assumed to be sigma-finite since the proof relies on the layer cake representation (see the erratum concerning page 26). It is also necessary to add a caveat to the uniqueness statement at the end of Theorem 1.14. The assumption is needed that the infimum, I, in (1), be finite.

Page 36. The last displayed equation in the proof of Corollary 1.19 should be changed to

$$g_\varepsilon(x) = \begin{cases} h_\varepsilon(x-a), & \text{if } x \le a+\varepsilon \\ 1, & \text{if } a+\varepsilon \le x \le b \\ h_\varepsilon(b+\varepsilon-x), & \text{if } x \ge b. \end{cases}$$

Page 48. In the displayed equation between (4) and (5): Replace $\lambda(y)$ by $\lambda(y)^p$.

Page 58. Our proof of Theorem 2.12 (Uniform boundedness principle) requires that the measure space be sigma-finite for the case $p = \infty$.

Page 65. Line 4 of Step 1 of the proof of Thm.2.16: In case j is not non-negative, we might not be able to take $C > 1$, but this does not matter. Thus, $C > 1$ should be replaced by $C > 0$.

Page 66. In the displayed equation between 2.16(4) and 2.16(5), H should be χ_H (twice). Also a subscript ϵ is missing for j.

Page 68. In the first line of the proof of Theorem 2.18, replace $\Omega \in \mathbb{R}^n$ by $\Omega \subset \mathbb{R}^n$.

Page 71. The line after the first displayed formula should read $\|f\|$ and not $\|f\|^2$.

Page 77. In Exercise 2.23 add the sentence: Here, Ω is any Lebesgue measurable subset of $\mathbb{R}^n$ of positive measure (which need not contain any ball).

Page 81. The line after formula (3) replace " ..two monotone functions.." by " ..two monotone non-decreasing functions.." . We also have to assume that $\phi_1(0)$ as well as $\phi_2(0)$ are zero, so that the layer cake representation Theorem 1.13 holds.

Page 81. In item (v) we have to require the function Φ to be lower semi-continuous since the left side is lower semi-continuous by definition. Since we defined f^* only for functions vanishing at infinity, for consistency, $\Phi \circ |f|$ must vanish at infinity, too, and this requires the two conditions that $\Phi(0) = 0$ and that Φ is continuous at 0. Also, just before (vi), change 'nonincreasing' to 'nondecreasing'.

Page 84, proof of 3.5. In equation (2) note that $J'_+(t) \geq 0$ and hence all the expressions in formula (2) are properly defined. Before the sentence beginning "A similar....." add the following note: (While there might be some ambiguity in (2) when $s = 0$, this is irrelevant since only a set of measure zero in $\mathbb{R}^n \times \mathbb{R}$ is involved.)

Page 95. In Exercise 3 A and B should be sets of finite measure.

Page 102. In the last line of equation (13) replace $\|g\|^q \|h\|^r$ by $\|g\|_q \|h\|_r$.

Page 107. After formula (5) it is mentioned that (3) and (5) are equivalent. The meaning of this statement is explained in Exercise 2 on page 121.

Page 112. The paragraph that begins "It is a theorem ..." should, instead, read "For $n \geq 3$, and for sufficiently regular transformations (C^4 will do), it is a theorem ...". The following sentence should be added to the end of the paragraph: "For $n = 2$ the conformal maps are analytic functions."

Page 121. In Exercise 4.2 change p to q.

Page 121. In Exercise 4.4 one has to assume that $\lambda > 0$.

Page 126. In 5.3(4) and the following equation, change $(x - y)^2$ to $|x - y|^2$.

Page 132. Delete the words "and for the Green's function of the Laplacian (before 6.20)" in the Remark at the top of the page.

Page143. Two lines before Section 6.9, the summation should be $j = 1$ to m and not $j = 1$ to n.

Pages 150-151. After the first sentence of 6.16 add the sentence: If (1) holds in $\mathcal{D}(\mathcal{O})$ then it holds for functions in $\mathcal{D}(\Omega)$ that have support in $\mathcal{O}$, and hence it holds for the whole of $\mathcal{D}(\Omega)$. On page 151 (4), and the line before, replace Ω by $\mathcal{O}$.

Page 153. In the top displayed equation, replace $|\nabla f(x)|^2$ by $|\nabla f(x)|$, i.e., remove the square from the right side.

Page 156. End of first paragraph after proof of Corollary 6.18: This sentence is correct, but it suffices to replace 'Ω is unbounded' by '$|\Omega|$ is infinite'.

Page 158. in the last equation there should be a '(' before '∂G_y'.

Page 173. In the proof of Theorem 7.3, $\mathbf{b}^m$ should be replaced by ∇f^m (twice).

Page 173. In Remark 1, the subscript on the inner product in the displayed equation should be $H^1(\Omega)$, not $H^1(\mathbb{R}^n)$.

Page 175. In the first displayed equation replace $\|f - g^m\|$ by $\|f - g^m\|^2$.

Page 180. In the middle of the page, the strong limit exists in L^p for $p < \infty$.

Page 184. In the second displayed formula replace μ by ν, i.e., the formula should read

$$K_\nu(z) \approx \frac{1}{2}\Gamma(\nu)\left(\frac{1}{2}z\right)^{-\nu}$$

Page184. Delete the limit sign in formula (1).

Page 189-190. The first sentence in the eighth line of the hypothesis of Theorem 7.17 should read: We define $(f, |p|f)$ here to be given by Eq. 7.12(4). We assume that f goes to 0 at ∞, but it is not assumed that $f \in L^2(\mathbb{R}^n)$.

The proof should start with the statement: Without loss of generality we can replace f by $|f|$, which does not change f^* or $|\nabla f|$ and only decreases $(f, |p|f)$ in 7.12(4). Thus, we assume, henceforth, that $f \geq 0$. We also take $1 > c > 0$ in part 1.

The assertion of the strictness of inequality (2) was not fully proved. The proof for $f \in L^2$ in part 2 was complete, but if $f \notin L^2$ the use of the approximation argument in part 1 conceivably could lose the strict inequality when the limit $c \to 0$ is taken. Here is a proof: It suffices to prove strictness for the quadratic form $Q(f, f) = (f, K_- f)$, where $(\phi, K_-\psi) := \int\int [\phi(x) - \phi(y)][\psi(x) - \psi(y)]K_-(x - y)dxdy$, instead of for $(f, |p|f) = (f, K_+ f) + (f, K_- f)$. Since $(f, K_+ f) \geq (f^*, K_+ f^*)$ it is not important if we lose strictness for $(f, K_+ f)$. Write $g(x) = (f(x) - c)_+$ and $h = f - g$. Clearly, $f = h + g$ and $f^* = h^* + g^*$. Then $(f, |p|f) = (f^*, |p|f^*) \Rightarrow Q(f, f) = Q(f^*, f^*) \Rightarrow Q(h, g) = Q(h^*, g^*)$ because $Q(g, g) \geq Q(g^*, g^*)$, etc. Now $Q(g, h)/2 = A - B$ with $A = \left(\int K_-\right)\left(\int g(x)h(x)dx\right)$.

Note that $\mathcal{I} := \int gh = c\int g$. The set $\{x : g(x) > 0\}$ has finite measure, but this set is not necessarily bounded. If it is bounded, then our assumption that $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ implies that $\mathcal{I} < \infty$, and we shall assume $\mathcal{I} < \infty$. (Note: If $n \geq 2$ then we can jump to the next chapter and use the Sobolev inequality, Theorem 8.1, to infer that $g \in L^{2n/(n-1)}(\mathbb{R}^n)$, and hence $\mathcal{I} < \infty$. A proof that $\mathcal{I} < \infty$ for all $n \geq 1$ is given in R. Frank and R. Seiringer, *Non-linear ground state representations and sharp Hardy inequalities*, arXiv: 0803.0503.)

By Theorem 3.9, the other term, $B = \int\int g(x)h(y)K_-(x-y)$, equals $\int\int g^*(x)h^*(y)K_-(x-y)$ only if $g = g^*$ and $h = h^*$, up to some common translation in $\mathbb{R}^n$. ■

Page 191. In the second line of Part 2 of the proof of 7.17, it should say $\|\nabla f\|_2^2 = \lim$ ...., i.e., the square is missing.

Page 191. End of PART 1. Remove the paragraph starting with "Because of the limit ... complete, however."

Page 191. End of PART 2, last paragraph: Change 'uniqueness' to 'strictness' (4 times). At the end, [Seiringer - Frank] should be [Frank - Seiringer].

Page 194. In the displayed equation in the middle of the page just above "and hence", replace the '-i' in the rightmost term by ' +', i.e, it should read $\chi_m(\nabla + iA)f + (\nabla\chi_m)f$.

Page 196. In Exercise 7 the inequality is in the wrong direction.

Page 202. In the expression on the last line of the statement of Theorem 8.3, $(x - a)$ should be $|x - a|$.

Page 205. In Equation (4) replace $\Gamma(n-1)$ by $\Gamma(N/2)$.

Page 207. In the last integral of the second displayed equation: the lower limit should be a and not x.

Page 210, Theorem 8.6. In the statement of the theorem replace "for $n = 1$ the convergence is pointwise and uniform" by "for $n = 1$ the convergence is pointwise and uniform on bounded sets".

Page 212. Second displayed equation, just after the line "note that by (4) and Hölder's inequality". Replace $2t$ by $4t$.

Page 213, Theorem 8.6. In the last paragraph of the proof, just after the first displayed eqn., replace "Moreover since $\|f^j\|_2$ is uniformly bounded, ...". by "Moreover, since $\|f^{j\prime}\|_2$ is uniformly bounded, ...".

Page 214. The "Remark" after Theorem 8.6, just before 8.7, is not accurate. First, the Remark should start as follows: It is worth noting, that for functions that are a-priori in some L^p space, e.g., in H^1, that statement (1)..... . Second, Theorem 8.4 should be replaced by Theorem 8.3.

Page 214. The inequality $0 < x_n < |x|\cos(\theta)$ in the second displayed expression is incorrect. It should be replaced by $x_n > |x|\cos(\theta)$.

Pages 216-217. In the second paraph the definition $g^j = (f^j - \varepsilon/2)_+$ is correct, but complicated. It suffices to define $g^j = f^j \geq 0$. At the bottom of page 216 replace H_0^1 by $W_0^{1,p}$ (twice). On the top of page 216 replace support by essential support (since f^j is not assumed to be continuous).

Page 218. In the line after equation (2) replace g^α by g_α.

Page 218. In Theorem 8.11 the case $p = 1$ is included in the statement. The Theorem as it stands is correct, but the proof given does not work in the case $p = 1$ since Theorem 2.18 requires $p > 1$.

To prove the Theorem for $p = 1$, we proceed as before, i.e., we assume that there exists a sequence of functions f^j such that $\|f^j - \int gf^j\|_q = 1$ for all j and that $\|\nabla f^j\|_1 \to 0$. Here q may be any number less than $n/(n-1)$ and hence $q > 1$. Next note that the sequence $h^j = f^j - \int gf^j$ has the same gradient as f^j and hence $\|\nabla h^j\|_1 \to 0$. By Theorem 2.18 there exists a function h in $L^q(\Omega)$ and a subsequence again denoted by h^j such that $h^j \rightharpoonup h$ weakly in $L^q(\Omega)$. For any function $\phi \in C_c^\infty(\Omega)$ we have that $\int h^j \nabla \phi = -\int \nabla h^j \phi$ and since $\|\nabla h^j\|_1 \to 0$, we learn that $\int h \nabla \phi = 0$ for all $\phi \in C_c^\infty(\Omega)$, i.e., $\nabla h = 0$ in the sense of distributions. Thus, by Theorem 6.11 h is constant. Clearly $0 = \lim_{j\to\infty} \int h^j g = \int hg$ and thus, $h = 0$. By the Rellich-Kondrachev theorem we know that the space $W^{1,1}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for all $q < n/(n-1)$. (Note that this formulation of the Rellich-Konrachev theorem is slightly more general then Theorem 8.9. For a proof, see e.g., Brézis.) Hence h^j converges strongly in $L^q(\Omega)$ and therefore $\|h\|_q = 1$ which is a contradiction. ■

Page 220. The constant in Nash's inequality is not correct. The factor

$$\left(1 + \frac{n}{2}\right)^{1+n/2} \qquad \text{should be replaced by} \qquad \left(1 + \frac{n}{2}\right)^{1+2/n}$$

Page 222. In paragraph 4 replace 'middle seventies by [Stam]' by 'late fifties by [Stam]'.

Page 230. After equation (3) remove the 'h' from Yoshida. It should read Yosida (but note that the 'si' is pronounced 'shi').

Page 232. Replace the rightmost T in eq. (1) by zero.
Pages 232-233. In Equation (2) of 8.18 and in the second displayed equation on page 233 replace $\|g_t\|_{p(t)}^{p(t)}$ by $\|g_t\|_{p(t)}^{p(t)-1}$.

In the sixth line of page 233 replace a by a^2, i.e., replace $a = 4\pi(p(t)-1)/(dp(t)/dt)$ by $a^2 = 4\pi(p(t)-1)/(dp(t)/dt)$.

In Equation (4) on page 233 replace $\|g_t\|_\infty$ by $\|g_T\|_\infty$.

Page 234. Theorem 8.18: Replace the rightmost t in eq. (6) by zero. In Eq. (1) replace $4\pi 0$ by $4\pi T$ and g_0 by f.

Page 235. Exercise 1 is not correctly stated. Consider three dimensions and take as a domain Ω the whole space without the origin. Removing one point does not change the Sobolev space, i.e., $H_0^1(\Omega) = H^1(\mathbb{R}^3)$. This example shows that the second assertion in the exercise is not correct. If one changes the problem by considering open and bounded sets then the second assertion is true. In fact it suffices to consider sets Ω whose complement is a set of positive capacity (see Section 11.15).

Page 240. In the third line replace 'in (2) become equalities' by 'in (3) become equalities'.

Page 250. The right side of equation (5) should have an overall factor $[(n-2)|\mathbb{S}^{n-1}|]^{-1}$, i.e., it should read

$$V(x) = \big[(n-2)|\mathbb{S}^{n-1}|\,\big]^{-1} \left[|x|^{2-n} \int_{|y|\le|x|} \mu(\mathrm{d}y) + \int_{|y|>|x|} |y|^{2-n} \mu(\mathrm{d}y) \right]$$

Page 251. Inequality (2) of Theorem 9.7 is correct only for $n \geq 3$, in which case our proof of it is correct. We failed to notice that the proof fails for $n < 3$, in which case the inequality is definitely false, as stated. It is only for $n \geq 3$ that $G_0(x) \to 0$ as $|x| \to \infty$ and this fact is the cause of the problem for $n < 3$.

Page 256. In Exercise 5(a) delete the sentence "You have to show ...". (The measurability is trivial.)

Page 259. Four lines after equation (4) replace " ... K_{f_2} and K_{f_1} have the same continuity and differentiability properties." by "... K_f and K_{f_1} have the same continuity and differentiability properties in B_1."

Page 268. In the limit sign in the last displayed equation on the page replace $j \rightharpoonup \infty$ by $j \to \infty$.

Page 271. In the first displayed equation change minus "-" to plus "+".

Page 271. In line 3 the potential $-|x|^{-3}$ is a bad example since it is not locally integrable. The potential $-|x|^{-5/2}$ will do the trick since the kinetic energy scales with λ^2 and the potential scales with $\lambda^{5/2}$, thus driving the energy to $-\infty$.

Page 272. In equation (6) change $(\psi, V\psi)$ to $|(\psi, V\psi)|$.

Page 274-275 The proof of 11.4 is correct but it can be streamlined. In the last line of p. 274 delete the words "for a sequence of ψ^j's,". On the top of p. 275, delete "there is a subsequence .... such that". The ψ^j converge weakly, by assumption, and Theorem 8.6 states strong convergence for the whole sequence. There is no need to pass to a subsequence.

Page 276. In Remark (2) replace C_0^∞ by C_c^∞.

Page 278. In the last paragraph "follows the integration by parts argument ..." should read "follows the lines of the integration by parts argument ..."

Page 281.. Change the end of the first paragraph from "strictly positive for all $x \in \mathbb{R}^n$" to "positive as in the definition in Remark (1) after the statement of Theorem 7.8 (Convexity inequality for gradients)."

Similarly, in the second paragraph replace "strictly positive" by "positive as in the definition in Remark (1) after the statement of Theorem 7.8.

The proof that ψ_0 in Theorem 11.8 is unique is not clearly stated. Insert the following at the end of the second paragraph on p.281.

If ψ_0 and ϕ_0 are minimizers then their real and imaginary parts are minimizers. We may therefore assume that both are real. Now $\chi_0 = \psi_0 + i\phi_0$ is also a minimizer and hence, up to a sign, is of the form $\chi_0 = (1 + ic)\psi_0$, where c is some constant and ψ_0 is positive. Hence $\phi_0 = c\psi_0$.

Page 301. In the third version of the min-max principle, the function ϕ appearing in formula (5) should be normalized, $(\phi, \phi) = 1$.

Page 306. Equation (4) it should read $n \geq 3$ instead of $n = 3$.

Page 318. Replace G_- by G_R in the first displayed equation.

Page 319. In Eq. (2) in Thm. 12.9 a square is missing on the left side, namely $|\widetilde{\psi}(k,y)|^2$

Page 326, 12.12(4). The statement and the proof are not right. Here is a correct upper bound and its proof: We start by trimming the negative part $[V]_-$ of the potential V. Let A be the subset of $\mathbb{R}^n$ where $V<0$ and consider the set A_L of all points in A whose distance to A^c is larger than a given number L. Define χ_L to be the characteristic function of the intersection of A_L with a cube of side length $1/L$. Thus, $\chi_L(x)V(x)\le 0$. Next we cut off the potential $\chi_L(x)[V]_-(x)$ at some large height, say h. Call this bounded potential $W_{L,h}$ and set $V_{L,h}=V_+-W_{L,h}$, where V_+ is the positive part of V. Since we made the negative part of the potential smaller, we have, by the min-max principle,

$$\sum_{j\ge 0} E_j(\mu V) \le \sum_{j\ge 0} E_j(\mu V_{L,h})$$

Choose $M_{L,h}(k,y)$ to be the characteristic function of the set $\{(k,y): p^2+\mu V_{L,h}<0\}=\{(k,y): p^2-\mu W_{L,h}<0\}$ and use the results on coherent states in Section 12.11 to obtain

$$\sum_{j\ge 0} E_j(\mu V_{L,h}) \le \int [|2\pi k|^2 + \mu V_{L,h} * G_R^2(x)] M_{L,h}(k,x)dkdx + C_n N_{L,h} n^2/R^2 \ ,$$

where $N_{L,h}=\int M_{L,h}(k,x)dkdx = (2\pi)^{-n}\frac{|\mathbb{S}^{n-1}|}{n}\int W_{L,h}^{n/2} \le (2\pi)^{-n}\frac{|\mathbb{S}^{n-1}|}{n}\mu^{n/2}h^{n/2}L^{-n}$. We also have that $\int [|2\pi k|^2+\mu V_{L,h}\star G_R^2(x)]M_{L,h}(k,x)dkdx =$

$$\mu^{n/2+1}(2\pi)^{-n}\frac{|\mathbb{S}^{n-1}|}{n+2}\int W_{L,h}^{n/2+1}dx + \mu^{n/2+1}(2\pi)^{-n}\frac{|\mathbb{S}^{n-1}|}{n}\int W_{L,h}^{n/2}V_{L,h} * G_R^2(x)dx \ . \qquad (\star)$$

Note, that for $R<L$ $\int W_{L,h}^{n/2}V_{L,h}*G_R^2(x)dx = \int\{G_R^2 * W_{L,h}^{n/2}\}V_{L,h}(x)dx = -\int\{G_R^2*W_{L,h}^{n/2}\}W_{L,h}dx$, because in this case, $\{G_R^2 * W_{L,h}^{n/2}\}(x)V_+(x)=0$. Now let $h,\mu\to\infty$ and $R<L\to 0$ such that $\mu^{-1}h^{n/2}n^2/L^nR^2\to 0$. We know, by monotone convergence, that $\int W_{L,h}^{n/2+1}dx \to \int V_-^{n/2+1}(x)dx$. Moreover, by adding and subtracting, we have $\int\{G_R^2\star W_{L,h}^{n/2}\}W_{L,h}dx - \int\{G_R^2\star V_-^{n/2}\}V_-dx = \int\{G_R^2*[W_{L,h}^{n/2}-V_-^{n/2}]\}W_{L,h}dx + \int\{G_R^2*V_-^{n/2}\}[W_{L,h}-V_-]dx$. By Hölder's inequality and Theorem 2.16

$$|\int\{G_R^2\star[W_{L,h}^{n/2}-V_-^{n/2}]\}W_{L,h}dx| \le \|W_{L,h}^{n/2}-V_-^{n/2}\|_{1+2/n}\|W_{L,h}\|_{1+n/2}$$

and, likewise, $|\int\{G_R^2*V_-^{n/2}\}[W_{L,h}-V_-]dx| \le \|V_-\|_{1+n/2}\|W_{L,h}-V_-\|_{1+n/2}$, and we see that both terms tend to zero as $h\to\infty$ and $L\to 0$. Using Theorem 2.16 once more we know that $G_R^2V_-^{n/2}$ converges strongly to $V_-^{n/2}$ in $L^{1+2/n}$. We conclude, from $(\star)$, that

$$\limsup_{\mu\to\infty}\mu^{-n/2-1}\sum_{j\ge 0}E_j(\mu V) \le -(2\pi)^{-n}\frac{2|\mathbb{S}^{n-1}|}{n(n+2)}\int V_-^{n/2+1}(x)dx \ .$$

Page 338. Change reference ‘Reed, M. and Simon, N.’ to ‘Reed, M. and Simon, B.’ .

Page 339. The reference to Hermann Weyl is not in alphabetical order.